



A tensegrity approach to the optimal reinforcement of masonry domes and vaults through fiber-reinforced composite materials



F. Fraternali^{a,*}, G. Carpentieri^{a,b}, M. Modano^b, F. Fabbrocino^c, R.E. Skelton^d

^a Department of Civil Engineering, University of Salerno, 84084 Fisciano, SA, Italy

^b Department of Structural Engineering, University of Naples Federico II, 80132 Naples, Italy

^c Department of Engineering, Pegaso University, Piazza Trieste e Trento, 48, 80132 Naples, Italy

^d Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA, 92093-0411, USA

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ABSTRACT

We present a tensegrity approach to the strengthening of masonry vaults and domes performed by bonding grids of fiber reinforced composites to the masonry substrate. A topology optimization of such a reinforcement technique is formulated, on accounting for a tensegrity model of the reinforced structure; a minimal mass design strategy; different yield strengths of the masonry struts and tensile composite reinforcements; and multiple loading conditions. We show that the given optimization strategy can be profitably employed to rationally design fiber-reinforced composite material reinforcements of existing or new masonry vaults and domes, making use of the safe theorem of limit analysis. A wide collection of numerical examples dealing with real-life masonry domes and vaults highlight the technical potential of the proposed approach.

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1. Introduction

The field of Discrete Element Modeling (DEM) of materials and structures is growing rapidly, attracting increasing attention from physicists and mechanicians working in different research areas. Originally, such a computational technique was aimed at describing particle interactions in discrete systems, via suitable force and/or torque systems (fully discrete systems, refer, e.g., to [1] and references therein). Nowadays, DEMs are also frequently used in association with continuous approximation schemes (coupled discrete-continuum models), in order to tackle scaling limitations of purely discrete models. DEMs may indeed require a large number of variables, being well suited to describe small process zones (dislocation and fracture nucleation, nanoindentation, atomic rearrangements, etc., cf. [2–7]).

In structural mechanics, a special class of DEMs is that of equivalent truss models of solids and structures, which includes Lumped Strain/Stress Models (LSM) of plates and shells [8–10]; Thrust Network Approaches (TNA) to masonry structures [11–16]; mechanical models of chains of granular materials or carbon nanotube (CNT) arrays [17,18]; and strut and tie models of discontinuous

regions in reinforced-concrete structures [19], just to name a few examples. Some convergence studies of such methods in the continuum limit are presented in [20–22] for bending plates, 2D elasticity, and CNT arrays, respectively.

Tensegrity structures are prestressable truss structures, which are obtained by stabilizing a set of compressed members (bars or struts) through a network of tensile elements (cables or strings). Tensegrity architectures have been used to describe a large variety of natural [23] and engineering systems [24–26], and it has been shown that the tensegrity approach to structural mechanics leads to design minimal mass systems in different mechanical problems [27–32].

The present work deals with the topology optimization of reinforcements of masonry vaults and domes realized through meshes of Fiber Reinforced Polymers (FRP) and/or Fabric Reinforced Cementitious Matrix (FRCM) composites bonded to the masonry substrate. We model the examined structures as tensegrity networks of masonry struts and tensile elements corresponding to the FRP/FRCM-reinforcements. Such reinforcements are often applied to masonry structures in the form of meshes of 1D elements [33,34], and are aimed at carrying tensile forces that would otherwise cause cracking damage of masonry [35–39]. The proposed optimization strategy determines the minimal mass tensegrity structure connecting a given node set, under prescribed yielding constraints. Each node is potentially connected to all the

* Corresponding author.

E-mail addresses: f.fraternali@unisa.it (F. Fraternali), gcarpentieri@unisa.it (G. Carpentieri), modano@unina.it (M. Modano), francesco.fabbrocino@unipegaso.it (F. Fabbrocino), bobskelton@ucsd.edu (R.E. Skelton).

neighbor nodes lying in a ball of given radius, through compressive and tensile elements. Such a connection pattern defines a background structure that is subject to minimal mass optimization [30], assuming different yield strengths for the masonry struts (compressive elements), and the FRP/FRCM reinforcements. An optimization procedure takes the node set defining the geometry of the structure (obtained, e.g., through a laser-scanner), the material density and the compressive and tensile material strengths as input parameters. It produces a minimal mass resisting mechanism of the reinforced structure as output, which can be regarded as a lumped stress/thrust network/strut and tie model of the examined structure [12,14,15]. Under the assumption of perfectly plastic response of masonry in compression and FRP/FRCM reinforcements in tension, the safe theorem of the limit analysis of elastic–plastic bodies [40] ensures that the reinforced structure is safe under the examined loading conditions. It is worth noting that the *Italian Guide for the Design and Construction of Externally Bonded FRP Systems for Strengthening Existing Structures* claims what follows: ‘Simplified schemes can also be used to describe the behavior of the structure. For example, provided that tensile stresses are directly taken by the FRP system, the stress level may be determined by adopting a simplified distribution of stresses that satisfies the equilibrium conditions but not necessarily the strain compatibility’ (see [39], Section 5.2.1). A minimal mass resisting mechanism allows for an optimized design of the FRP-/FRCM-reinforcements, preventing excessive over-strength of the reinforced structure, which may be responsible for reduced ‘cracking-adaptation’ capacity [41].

It worth noting that the strengthening of pre-existing masonry structures may require the application of a suitable state of pre-stress to be effective [39].

The paper is structured as follow. Section 2 describes the proposed tensegrity model of a reinforced masonry vault or dome, which is based on an automatically generated background structure. Next, Section 3 formulates a minimum mass optimization of such a structure, under given yielding constraints and multiple loading conditions. The following Section 4 presents a parade of case studies dealing with FRP-/FRCM-reinforcements of a dome (Section 4.1), a groin vault (or cross vault), a cloister vault (or domical vault) and a barrel vault (Section 4.2). Concluding remarks and prospective work are illustrated in Section 5.

2. Tensegrity model of a reinforced masonry vault

Let us consider a masonry vault or dome with mean surface described by a set of n_n nodes in the 3D Euclidean space. In a given Cartesian frame $\{O, x, y, z\}$, the components (x_k, y_k, z_k) of the position vectors \mathbf{n}_k of all such nodes ($k = 1, \dots, n_n$) can be arranged into the following $3 \times n_n$ node matrix

$$\mathbf{N} = \begin{bmatrix} x_1 & \dots & x_{n_n} \\ y_1 & \dots & y_{n_n} \\ z_1 & \dots & z_{n_n} \end{bmatrix} \quad (1)$$

We now introduce a *background structure*, which is obtained by connecting each node \mathbf{n}_k with all the neighbors \mathbf{n}_j such that it results $|\mathbf{n}_k - \mathbf{n}_j| \leq r_k$ (*interacting neighbors*). Here, $|\mathbf{n}_k - \mathbf{n}_j|$ is the Euclidean distance between \mathbf{n}_k and \mathbf{n}_j , and r_k is a given *connection radius*. Fig. 1 shows the particular case in which the interacting neighbors of a selected node coincide with its nearest neighbors. We connect \mathbf{n}_k to each interacting neighbor \mathbf{n}_j through two elements working in parallel: a compressive masonry strut (or *bar*) $\mathbf{b}_i = \mathbf{n}_k - \mathbf{n}_j$, and a tensile FRP/FRCM element (or *string*) $\mathbf{s}_i = \mathbf{n}_k - \mathbf{n}_j$. The minimal mass optimization of the background structure presented in Section 3 will choose which one such members (bar or string) is eventually present between nodes \mathbf{n}_k and \mathbf{n}_j

in the optimized configuration (i.e., which one of the above members eventually carries a nonzero axial force in the minimal mass configuration, see also [30], Section 7). For future use, we let n_b and n_s denote the total number of bars and the total number of strings composing the background structure, respectively (with $n_b = n_s$ in the non-optimal configuration), and we set $n_x = n_b + n_s$.

We assume that the background structure is subject to a number m of different loading conditions, and, with reference to the j -th condition, we let $\lambda_{b_i}^{(j)}$ denote the compressive force per unit length (force density) acting in the i -th bar, and let $\gamma_{s_i}^{(j)}$ denote the tensile force per unit length acting in the i -th string, both defined to be positive quantities. The static equilibrium equations of the nodes in correspondence of the current load condition can be written as follows

$$\mathbf{A}\mathbf{x}^{(j)} = \mathbf{w}^{(j)} \quad (2)$$

where \mathbf{A} is the $3n_n \times n_x$ static matrix of the structure, depending on the geometry and the connectivity of bars and strings (see [30]); $\mathbf{w}^{(j)}$ is external load vector, which stacks the $3n_n$ Cartesian components of the external forces acting on all nodes in the current loading condition; and $\mathbf{x}^{(j)}$ is the vector with n_x entries that collects the force densities in bars and strings in correspondence of the same loading condition, that is

$$\mathbf{x}^{(j)} = [\lambda_1^{(j)} \dots \lambda_{n_b}^{(j)} \gamma_1^{(j)} \dots \gamma_{n_s}^{(j)}]^T \quad (3)$$

Let σ_{b_i} and σ_{s_i} respectively denote the compressive strength of the generic bar and the tensile strength of the generic string forming the background structure, which we hereafter assume behaving as perfectly plastic members. Yielding constraints in bars and strings require that, for each loading condition, it results

$$\lambda_i^{(j)} b_i \leq \sigma_{b_i} A_{b_i}, \quad \gamma_i^{(j)} s_i \leq \sigma_{s_i} A_{s_i} \quad (5)$$

where A_{b_i} and A_{s_i} respectively denote the cross-section areas of the generic bar and string.

The masses of the generic bar and string of the background structure are computed as follows

$$m_{b_i} = \rho_{b_i} A_{b_i} b_i, \quad m_{s_i} = \rho_{s_i} A_{s_i} s_i, \quad (6)$$

where ρ_{b_i} and ρ_{s_i} denote the mass densities of such members, respectively.

3. Minimal mass design

Following [30], we formulate a *minimal mass design* of the background structure through the following linear program

$$\begin{aligned} & \underset{\mathbf{x}^{(j)}, \mathbf{y}}{\text{minimize}} && m = \mathbf{d}^T \mathbf{y} \\ & \text{subject to} && \begin{cases} \mathbf{A}\mathbf{x}^{(j)} = \mathbf{w}^{(j)} \\ \mathbf{C}\mathbf{x}^{(j)} \leq \mathbf{D}\mathbf{y} \\ \mathbf{x}^{(j)} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \end{cases} \end{aligned} \quad (7)$$

where

$$\mathbf{y} = [A_{b_1} \dots A_{b_{n_b}} | A_{s_1} \dots A_{s_{n_s}}]^T \quad (8)$$

$$\mathbf{d}^T = [\varrho_{b_1} b_1 \dots \varrho_{b_{n_b}} b_{n_b} | \varrho_{s_1} s_1 \dots \varrho_{s_{n_s}} s_{n_s}] \quad (9)$$

$$\mathbf{C} = \begin{bmatrix} \text{diag}(b_1, \dots, b_{n_b}) & \mathbf{0} \\ \mathbf{0} & \text{diag}(s_1, \dots, s_{n_s}) \end{bmatrix} \quad (10)$$

$$\mathbf{D} = \begin{bmatrix} \text{diag}(\sigma_{b_1}, \dots, \sigma_{b_{n_b}}) & \mathbf{0} \\ \mathbf{0} & \text{diag}(\sigma_{s_1}, \dots, \sigma_{s_{n_s}}) \end{bmatrix} \quad (11)$$

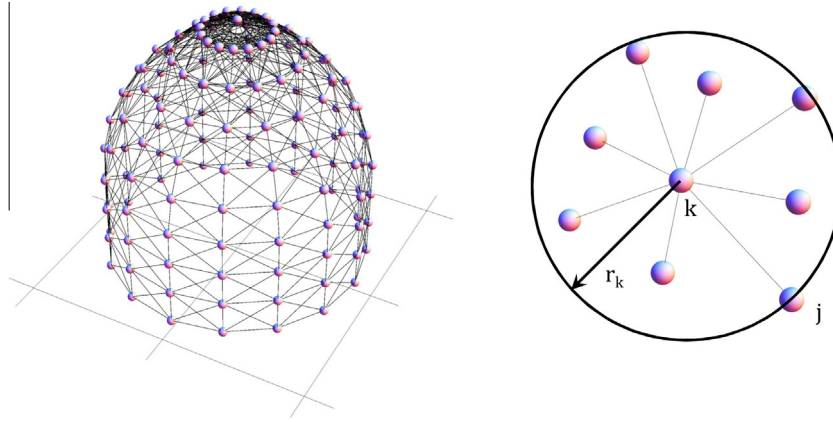


Fig. 1. Background structure associated with a node set extracted from a dome (left) and interacting neighbors of a selected node (right).

Problem (7) returns an optimal topology of the background structure, which consists of the set of all members (bars and strings) exhibiting nonzero force density ($\lambda_i^{(j)}$ or $\gamma_i^{(j)}$) in at least one of the examined loading conditions. The optimal configuration exhibits minimal mass among all the possible configurations of the background structure, under the equilibrium constraints (2) and the yielding constraints (5). It is worth noting that the mass of the background structure should not be confused with the self-weight of the masonry dome or vault under examination, which we agree to include in the external load vector $\mathbf{w}^{(j)}$. The quantity subject to minimization in problem (7) should instead be regarded as the mass of an internal resisting mechanism of the structure. As we already observed, the latter is formed by a collection of masonry struts (bars), and a network of FRP/FRCM reinforcements loaded in tension (strings), which are able to carry axial forces that equilibrate the examined external loads without violations of the local yielding constraints.

4. Numerical results

This section presents a parade of applicative examples of the optimization procedure formulated in Section 3, which deal with the FRP/FRCM reinforcements of a masonry dome (Section 4.1), and three different typologies of masonry vaults: a groin vault (or cross vault), a cloister vault (or domical vault) and a barrel vault (Section 4.2). Let $\{O, x, y, z\}$ be a Cartesian frame with the z -axis oriented upward along the vertical direction. We consider a vertical loading condition, corresponding to the masonry self-weight, and four loading conditions summing the effects of the masonry self-weight with those produced by horizontal forces acting along the $+x, -x, +y$, and $-y$ directions. The latter are equal in magnitude to 35% of the vertical forces, and mimic the effects of seismic excitations of the examined structures, through a conventional, static approach (refer, e.g., to Eurocode 8 [42]). We name *seismic loadings* the conditions that sum the masonry self weight to the above horizontal forces. The examined masonry consists of a ‘Neapolitan’ tuff brick masonry, which is largely diffused in the area of Naples. We assume that such a masonry exhibits 15.0 kN/m^3 self-weight, and 13 MPa compressive strength σ_b [44]. In addition, we assume a tensile strength σ_s equal to 376.13 MPa in the string elements, on taking an average value of the bond strengths of the FRP and FRCM reinforcements of masonry structures analyzed in [33,34], respectively. The examined background structures are restrained by fixed hinge supports at the basis. We used the software Tensopt [43] to numerically solve problem (7).

4.1. The dome of the church of Santa Maria di Monteverginella in Naples

Let us analyze the dome of the church of Santa Maria di Monteverginella in Naples, whose FRP-/FRCM-reinforcement has already been studied in [44] through a r -adaptive finite-element approach (refer to Fig. 2 for geometric details). We model the middle surface of such a dome through a tensegrity network with 145 nodes and 1504 connections (background structure in Fig. 1). The optimal FRP/FRCM reinforcement patterns obtained through the procedure formulated in Section 3 are shown in Fig. 3a–i.

Under vertical loading, the results in Fig. 3a–c highlight that the minimal mass FRP/FRCM reinforcements of the current structure are polar-symmetric and placed along parallel circles above the drum, with width increasing downward. Seismic loading in the x direction instead combines parallel-circles’ reinforcements with diagonal reinforcements placed over the portions of the dome parallel to the x - z plane ($\pm y$ edges, cf. Fig. 3d–f). In this case, the widths of the reinforcements placed over the $+x$ edge and the $\pm y$ edges of the dome are considerably larger than the widths of the reinforcements placed over the $-x$ edge.

The ‘combo’ loading condition combining vertical loading and seismic loading in two perpendicular directions returns polar-symmetric and parallel-circle reinforcements (Fig. 3g–i). It is worth noting that the FRP/FRCM reinforcements corresponding to seismic loading have markedly larger widths than those corresponding to vertical loading (compare Fig. 3a–c with Fig. 3g–i). The width of the top hoop reinforcements under vertical loading is about 100 mm (cf. Fig. 3a–c), while that of the central hoop reinforcements

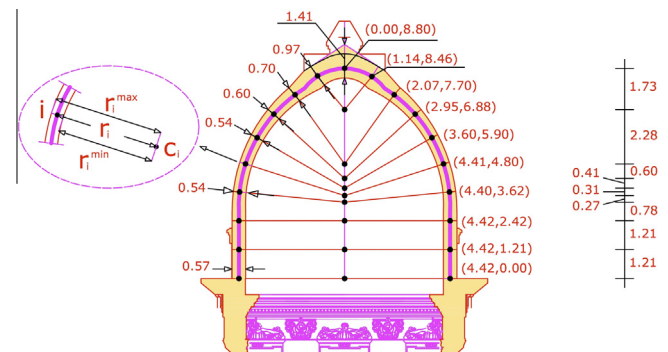


Fig. 2. Cross-section of the dome of the church of Santa Maria di Monteverginella in Naples.

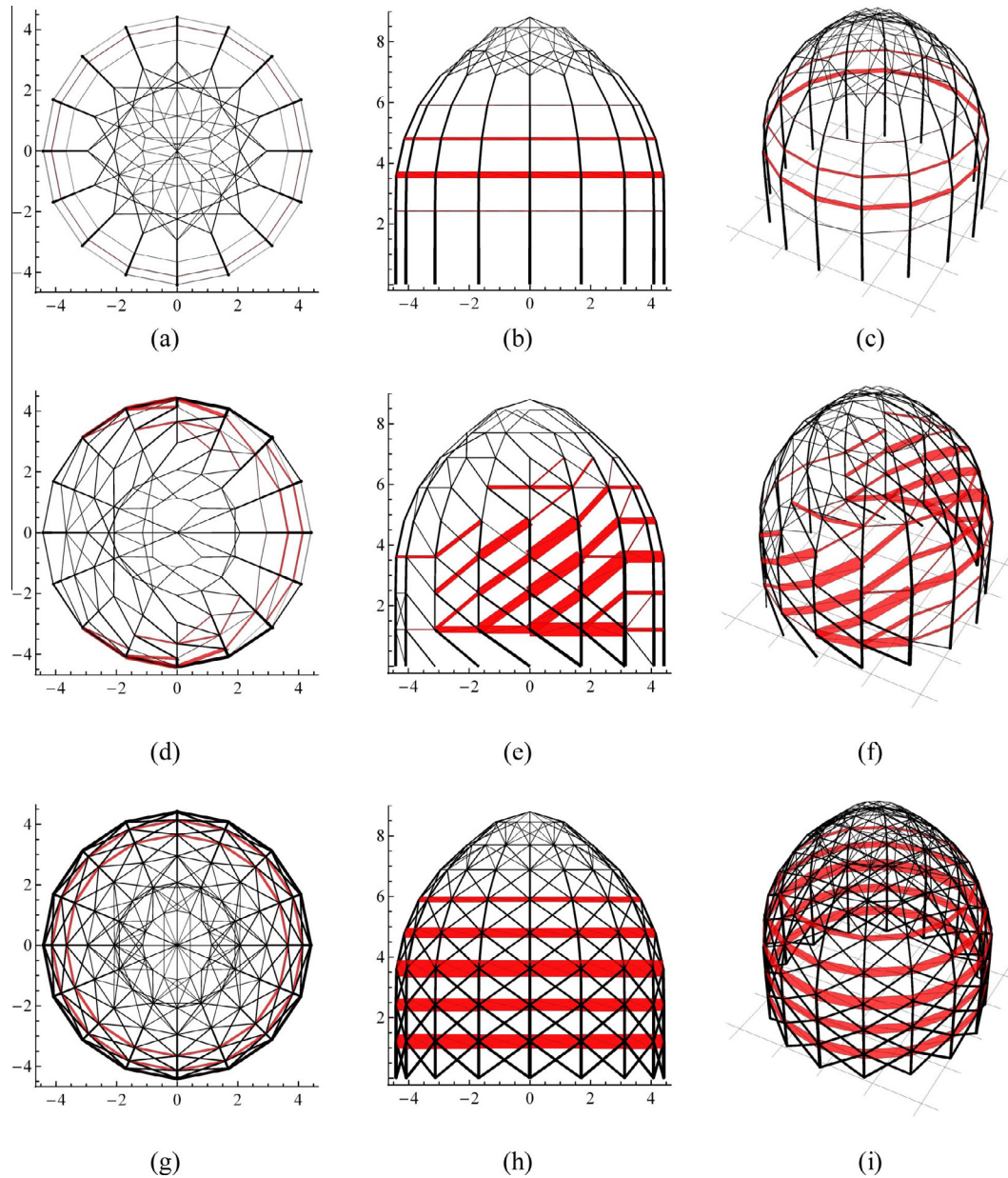


Fig. 3. Top, side and 3D views of the optimal reinforcement patterns of a masonry dome with FRP/FRCM strips of 0.34 mm thickness (marked in red), under different loading conditions. (a)–(c): Vertical loading. (d)–(f): Seismic loading in the +x-direction. (g)–(i): Combined vertical loading and seismic loading in two perpendicular directions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

under combo vertical and seismic loading is 578 mm (cf. Fig. 3g–i). The compressed members of the internal resisting mechanism of the dome are mainly composed of meridian-shaped struts (Fig. 3a–c), which are combined with diagonal struts in presence of seismic loading (Fig. 3d–i). The results in Fig. 3 are in agreement with the frequent observations of ‘meridional’ (or ‘orange-slice’) crack patterns in unreinforced masonry domes (refer, e.g., to [41,44] and references therein).

4.2. Groin, cloister and barrel vaults

Figs. 4–6 show the minimal mass FRP/FRCM reinforcements that we obtained for a groin vault, a cloister vault and a barrel vault, respectively. The geometries of the examined vaults are illustrated in above figures, together with the corresponding background structures.

The background structure of the examined groin vault features 237 nodes and 1840 connections (cf. Fig. 4a–c). The minimal mass reinforcement pattern of such a vault consists of FRP/FRCM strips with 0.17 mm thickness on the web panels (width of the meridian strips near the crown under vertical loading: 340 mm; total width of the square reinforcing patch covering the crown under combo seismic loading: 3000 mm); and 200 mm × 3.24 mm FRCM strips by the side of the groins at the corners (Fig. 4). The latter can be eventually replaced with pultruded FRP profiles with circular cross-section, 11.18 mm radius and 620.5 MPa tensile strength [45]. It is worth observing that the above reinforcements prevent ‘hinging’ cracks departing from the crown and meridian cracks, in the case of vertical loading (Fig. 4d–f); and meridian cracks, cracks parallel to the groins, and the so-called ‘Sabouret’ cracks parallel to wall ribs, under combined vertical and seismic loading (Fig. 4g–i). The masonry strut network of the groin vault consist

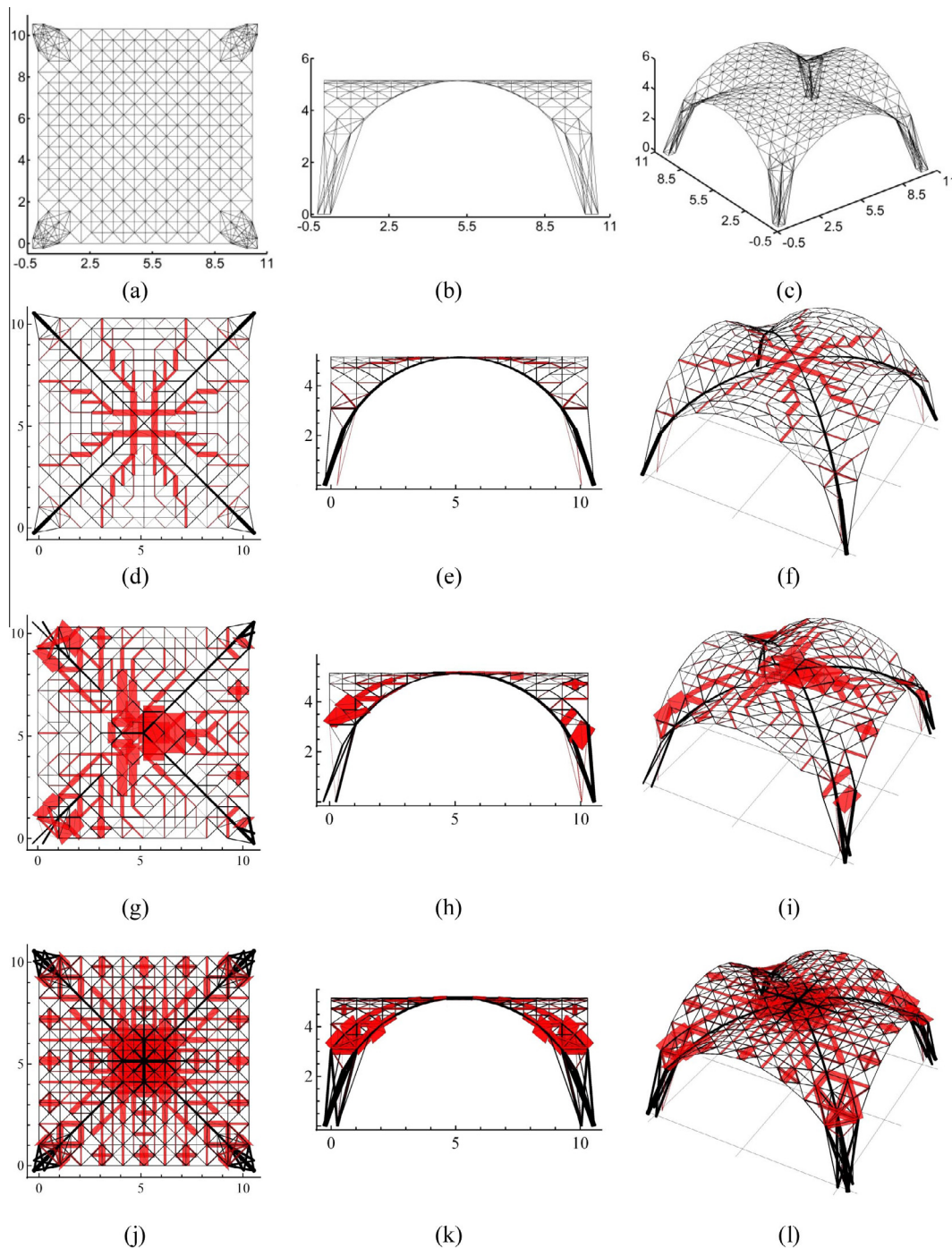


Fig. 4. Top, side and 3D views of the optimal reinforcement patterns of a groin vault with FRP/FRCM strips of 0.17 mm thickness on the web panels, combined with 200 mm × 3.24 mm FRCM strips or, alternatively, 11.18 mm radius pultruded FRP profiles at the corners (reinforcements marked in red). (a)–(c): Background structure. (d)–(f): Vertical loading. (g)–(i): Seismic loading in the +x-direction. (j)–(l): Combined vertical loading and seismic loading in two perpendicular directions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

of four main arches at the ribs, which are completed by secondary meridian arches and diagonal struts over the webs.

For the cloister vault we employed a background structure with 441 nodes and 4508 connections (see Fig. 5a–c). The optimal reinforcement of such a vault under vertical loading is mainly formed by parallel FRP/FRCM strips with 0.17 mm thickness and 82 mm maximum width near the crown (Fig. 5d–f). The above reinforcements are combined with diagonal FRP/FRCM strips with about 140 mm maximum width near the intersections of the four vault segments, under combined vertical and seismic loading (Fig.

5g–l). The compressed truss network include couples of diagonal arches near the corners, parallel-line arches, and diagonal struts over the vault segments (Fig. 5d–l).

In the case of the barrel vault, we focused our attention on seismic loading in the +x direction (Fig. 6a–c), and combined vertical loading and seismic loading in two orthogonal directions (Fig. 6d–f), neglecting simple vertical loading. The background structure of the current vault is composed of 231 nodes and 1660 connections (Fig. 6). The resisting mechanism of the barrel vault under seismic loading includes transverse compressed

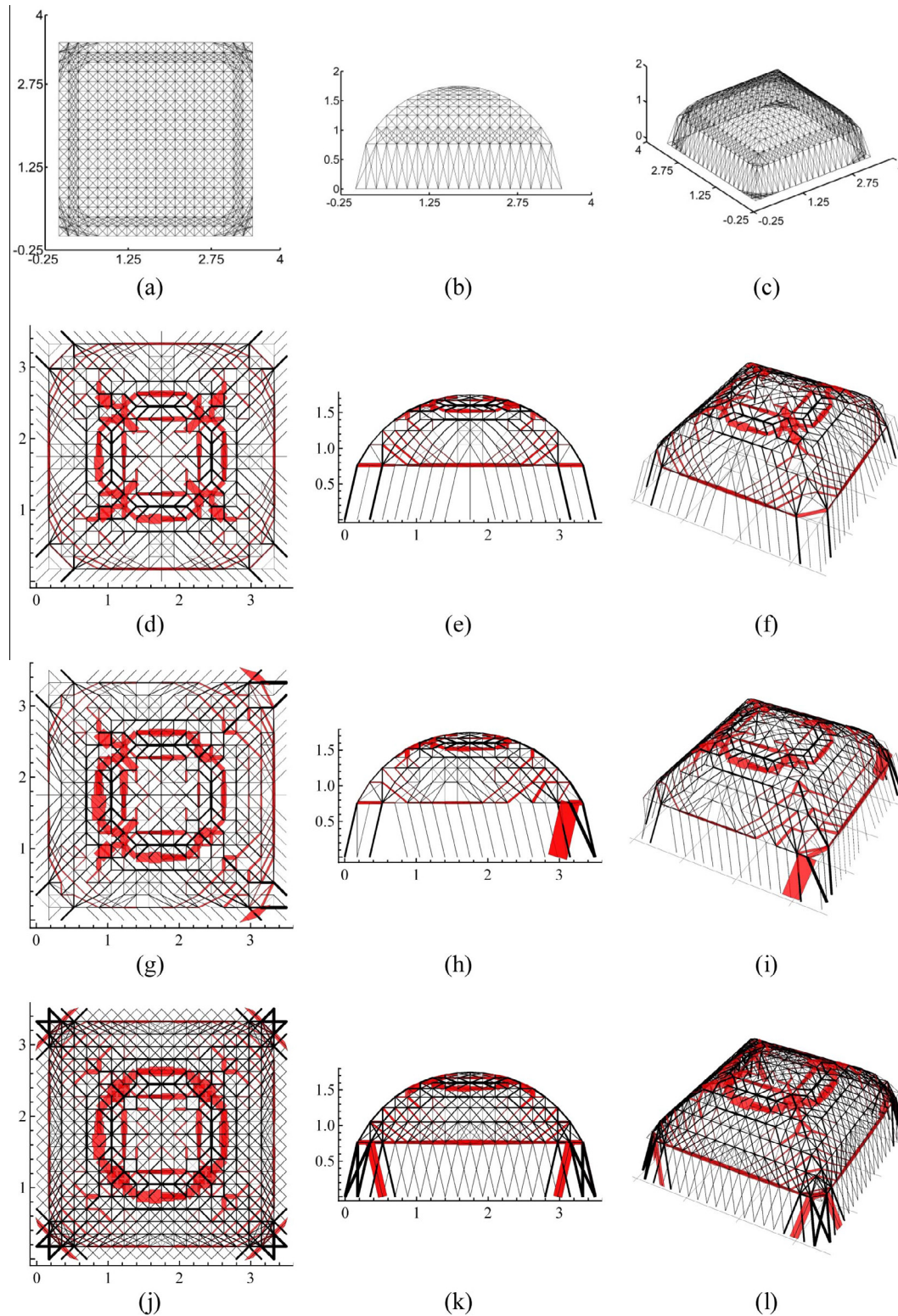


Fig. 5. Top, side and 3D views of the optimal reinforcement patterns of a cloister vault with FRP/FRCM strips of 0.17 mm thickness (marked in red), under different loading conditions. The widths of the FRP/FRCM reinforcements are magnified by a factor 2 for visual clarity. (a)–(c): Background structure. (d)–(f): Vertical loading. (g)–(i): Seismic loading in the +x-direction. (j)–(l): Combined vertical loading and seismic loading in two perpendicular directions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

arches, longitudinal FRP/FRCM strips with 0.17 mm thickness, and diagonal struts, as shown in Fig. 6d–f. The FRP/FRCM reinforcements feature rather small width in the present case (minimum width: 0.1 mm; maximum width: 4 mm), and have been magnified by a factor 10 in Fig. 6 for visual clarity.

5. Concluding remarks

We have presented a minimal mass approach to the search for internal resisting mechanisms of masonry domes and vaults composed of compressed masonry struts and tensile FRP/FRCM

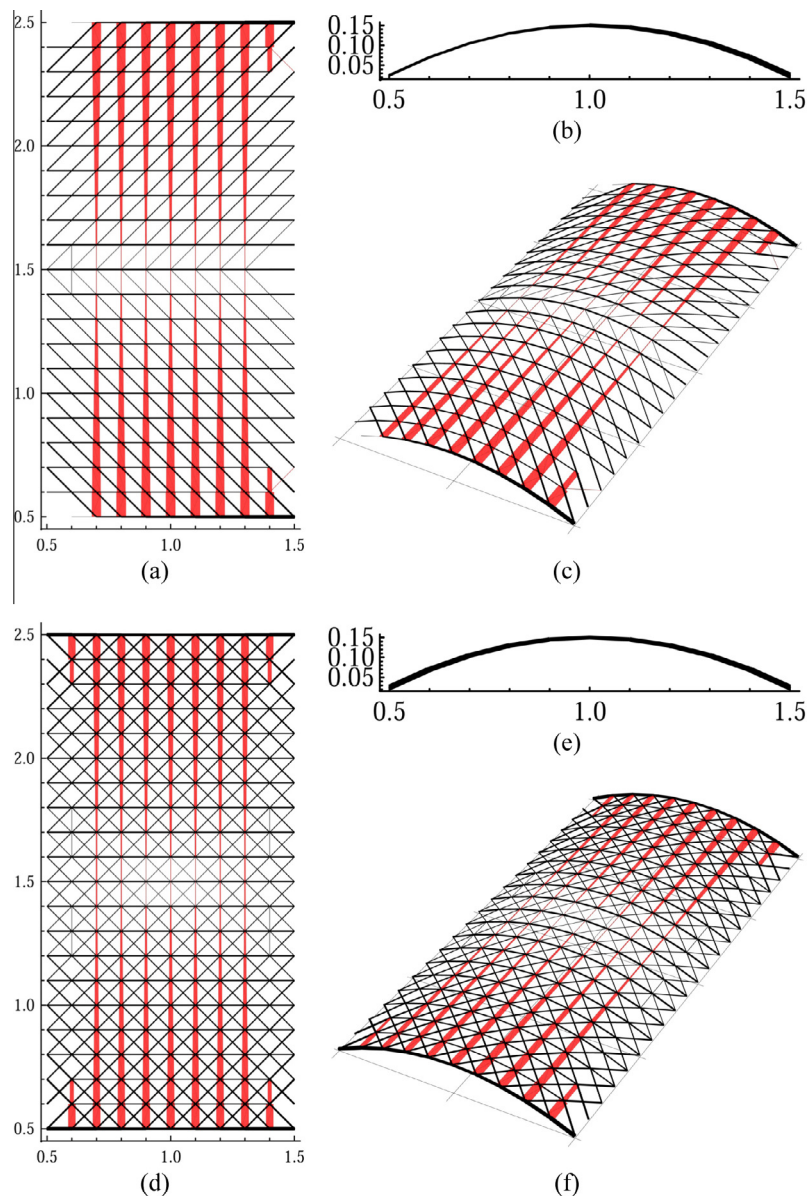


Fig. 6. Top, side and 3D views of the optimal reinforcement patterns of a barrel vault with FRP/FRCM strips of 0.17 mm thickness (marked in red). The widths of the FRP/FRCM reinforcements are magnified by a factor 10 for visual clarity. (a)–(c): Seismic loading in the +x-direction. (d)–(f): Combined vertical loading and seismic loading in two perpendicular directions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

reinforcements. Such mechanisms can be regarded as tensegrity/strut and tie models of the examined structures, in line with available technical standard for the FRP reinforcement of masonry structures [39]. The existence of internal resisting mechanisms subject to given yielding constraints ensures that the reinforced structure is safe under the examined loading conditions, according to the safe theorem of the limit analysis of perfectly plastic bodies [40]. Several numerical examples have highlighted the potential of the proposed approach in designing minimal mass FRP/FRCM reinforcements of masonry vaults and domes, which are aimed at preserving sufficient ‘cracking-adaptation’ capacity of the reinforced structure.

Future directions of the present study will be devoted to formulating tensegrity models of FRP/FRCM reinforced planar masonry structures (e.g., masonry walls), and three-dimensional assemblies of domes, vaults and supporting structures (including walls, piers, flying buttresses; drums; etc.). Additional extensions of the present research will deal with discrete-to-continuum approaches to tensegrity membranes [6], and 3D tensegrity networks.

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