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A mixed lumped stress–displacement approach to the elastic problem of masonry walls

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ABSTRACT

We present a novel approach to the elastic problem of masonry walls, which generalizes the lumped stress method presented in Fraternali (2001, 2007, 2010) and Fraternali et al. (2002). The generalization consists of a mixed lumped stress–displacement approach to the elastic problem of a wall that incorporates no-tension elements. Such an approach depends on the nodal values of the Airy stress function and the displacements of selected ("pivot") nodes. The latter coincide with inter-element and boundary nodes. The mixed lumped stress–displacement method can be conveniently coupled with standard finite element and boundary element approaches. Numerical applications dealing with recurrent structural elements are given, showing that such a method is able to capture some essential features of the actual response of masonry constructions.

1. Introduction

It is well known that the numerical implementation of the elastic no-tension (ENT) model of masonry structures may involve a number of operational difficulties, such as non-existence, indeterminacy and/or discontinuity of the solution, divergence or locking of the numerical approximations in the limit of the mesh size tending to zero, etc. This is mainly due to the "sharp" nature of the ENT constitutive equations (cf., e.g., Giaquinta and Giusti, 1989; Del Piero, 1989). Stress approaches to ENT (or "masonry-like") structures exhibit some peculiar advantages, since one can prove the uniqueness of the solution of the ENT boundary value problem in terms of the stress field (Cuomo and Ventura, 2000; Angelillo, 1994; Fraternali, 2007; Fortunato, 2000). Interesting displacement approaches for no-tension bodies have been proposed in Lucchesi et al. (1994), Baratta and Corbi (2004), and Angelillo et al. (2010).

A lumped stress method (LSM) for the elastic problem of a plane body Ω has been formulated in Fraternali (2001, 2007), and Fraternali et al. (2002). Such a non-conforming method approximates the 2D stress field through a piece-wise constant regularization of singular (or lumped) stresses, which are defined over the skeleton of a triangulation Π_h of Ω. It leads to approximate the continuous body with a non-conventional truss structure, whose energy is defined per nodes (and *not* per elements) over a dual tessellation $\hat{\Pi}_h$ of Ω. The convergence of the LSM towards the "exact" solution of the continuous boundary value problem has been proved mathematically in Fraternali (2007), and numerically in Fraternali (2001) and Fraternali et al. (2002). Its application to plane ENT bodies and vaulted masonry structures has been presented in Fraternali (2007, 2010), respectively, through a pure stress approach. A similar representation of the stress field of masonry structures is at the basis of the thrust network analysis (TNA) presented in Block and Ochsendorf (2007) and Block (2009).

The present paper deals with a mixed LSM-displacement method (LSDM) for the elastic problem of a wall that incorporates ENT elements. As in the standard LSM, the stress field of the wall is approximated through a network of lumped stresses, making use of polyhedral stress functions and a complementary energy approach (Fraternali, 2001; Fraternali et al., 2002). The standard LSM admits either a stress function formulation (cf. Fraternali, 2001, 2007; Fraternali et al., 2002), or a nodal displacement formulation (Fraternali et al., 2002). The first one (SFF) consists of a cost-effective equilibrium approach, which assumes the nodal values of the stress function as primary unknowns (just one degree of freedom per node). Nevertheless, the SFF is restricted to simple connected bodies, does not allow for the direct computation of nodal displacements, and is not easily coupled with other numerical methods, such as finite elements and boundary elements approaches. The nodal displacement formulation (NDF), on the other hand, leads to a full-range stiffness matrix. The originality of the present LSDM approach consists of a special treatment of the inter-element equilibrium equations of the discrete model, which are approached through an augmented Lagrangian technique (Nocedal and Wright, 2006). This leads us to a mixed approach, which admits the nodal values of the polyhedral stress functions of the different wall elements and the displacements

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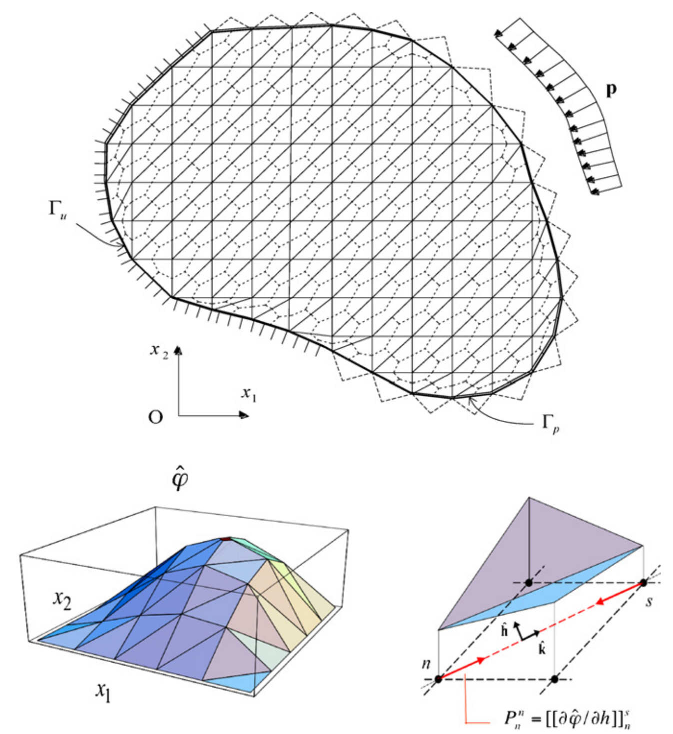

Fig. 1. (Top) Primary and secondary meshes of a plane body described through the lumped stress method. (Bottom) Polyhedral approximation of the Airy stress function and current lumped stress P_n^s.

of “pivot” nodes as primary unknowns. In particular, the stiffness matrix ruling the update of pivot displacements is diagonal. The no-tension constraint is traced back to the concavity of the polyhedral stress functions (Giaquinta and Giusti, 1989), and use is made of the convex-hull technique (Avis and Fukuda, 1992) to formulate appropriate initial points of the solution-search strategy (Angelillo and Rosso, 1995).

The structure of the paper is as follows. We begin by summarizing the main ingredients of the LSM in Section 2. Next, we present the mixed LSM-displacement method (LSDM) in Section 3, and some numerical results in Section 4. Finally, we describe the main conclusions of the present study and future work in Section 5.

2. Preliminaries on the lumped stress method

Let us consider the elastic problem of a plane body Ω subject to kinematical boundary conditions $\mathbf{u} = \bar{\mathbf{u}}$ on a given portion Γ_u of its boundary $\Gamma \equiv \partial\Omega$, and surface tractions $\mathbf{p}$ over $\Gamma_p = \Gamma/\Gamma_u$. We suppose, for the sake of simplicity, that Ω is polygonal and simply connected and that and no body forces are applied. The first assumption allows us to exactly cover Ω with a triangulation $\Pi_h = \{\Omega_1, \ldots, \Omega_M\}$ (*primary mesh*), and a dual tessellation $\hat{\Pi}_h = \{\hat{\Omega}_1, \ldots, \hat{\Omega}_N\}$ (*dual mesh*). We extend Π_h outside the portion Γ_p of Γ, introducing an “extended mesh” Π'_h (Fig. 1). Here and in what follows, $h = \sup_{m \in \{1,\ldots,M\}} \{\mathrm{diam}(\Omega_m)\}$ denotes the *mesh size*.

The assumptions of zero body forces and simple-connectivity of Ω allow us to derive the stress field of the body from a single-valued scalar potential or Airy stress function φ (cf., e.g., Gurtin, 1972). The *lumped stress method* presented in Fraternali (2001) and Fraternali et al. (2002) approximates φ through piece-wise linear functions $\hat{\varphi}$ defined over Π'_h (Fig. 1), and introduces the following “relaxed” version of the complementary energy of the body

$$E_h(\hat{\varphi}) = \frac{1}{2}\sum_{n=1}^{N}\sum_{s,t=1}^{S_n}\hat{A}_n^{st} P_n^s(\hat{\varphi}) P_n^t(\hat{\varphi}) - \sum_{n \in U}\mathbf{R}_n(\hat{\varphi}) \cdot \bar{\mathbf{u}}_n, \tag{1}$$

where

$$\hat{A}_n^{st} = \frac{\ell_n^s \ell_n^t A[\hat{\mathbf{h}}_n^s \otimes \hat{\mathbf{h}}_n^s] \cdot \hat{\mathbf{h}}_n^t \otimes \hat{\mathbf{h}}_n^t}{4\,|\hat{\Omega}_n|} \tag{2}$$

In (1) and (2), N is the total number of nodes of Π_h; S_n indicates the number of nearest neighbors of the generic node n; ℓ_n^s is the length of the edge $n-s$; $\hat{\mathbf{k}}_n^s$ and $\hat{\mathbf{h}}_n^s$ are the tangent and normal unit vectors to such an edge, respectively; $P_n^n = [[\partial\hat{\varphi}/\partial h]]_n^s$ is the jump of $\nabla\hat{\varphi} \cdot \hat{\mathbf{h}}_n^s$ across $n-s$ (i.e., the normal derivative of $\hat{\varphi}$ through this edge); and it results

$$R_n(\hat{\varphi}) = -\sum_{s=1}^{S_n} P_n^s(\hat{\varphi})\hat{\mathbf{k}}_n^s \tag{3}$$

The quantities P_n^s can be regarded as the axial forces carried by the bars of an *ideal truss* B_h, which has the same geometry of the skeleton of Π'_h (Fig. 1). Similarly, the quantity $\mathbf{R}_n$ can be regarded as the total force acting at node n of such a truss. Due to the assumption of zero body forces, $\mathbf{R}_n$ will be nonzero only at the boundary (support reaction). The discrete functional (1) defines a non-conventional complementary energy of the truss B_h, which is defined per dual elements $\hat{\Omega}_n$, and not per elements (as in an ordinary truss).

Let φ_o denote the minimizer of the “exact” complementary energy of the body, and $\hat{\varphi}_h$ the minimizer of (1). It has been shown in Fraternali (2007) that $\hat{\varphi}_h$ strongly converges to φ_o in the limit $h \to 0$, under suitable smoothness assumptions on φ_0 and the primal and dual meshes. The latter require that Π_h has a structured core, and $\hat{\Pi}_h$ is made up of polygons connecting the middle points of the edges of Π_h with the barycenters of the primal triangles (“barycentric” dual mesh, cf. Fig. 1). A Γ-convergence proof of the LSM for the biharmonic problem of isotropic elasticity is given in Davini (2002), considering families of triangulations that are regular in the sense of Ciarlet (1978). Such meshes include unstructured coverings of Ω. A strong/weak convergence proof of the LSM for ENT bodies and distorted triangulations is an open and challenging problem, which is beyond the scopes of the present work. We hereafter follow Fraternali et al. (2002) assuming that the dual mesh is barycentric.

3. A mixed LSM-displacement approach to masonry walls

We now examine a wall made up of an arbitrary collection of simply connected elements (beams, columns, etc.) $\Omega_1, \ldots, \Omega_{N_e}$, which include either standard elastic elements and/or ENT elements. As we shall see in short, such a discretization of the wall allows us to conveniently combine the SFF and NDF formulations of the LSM presented in Fraternali et al. (2002). We name *pivot* the interelement and boundary nodes that are not subject to kinematical constraints. On applying the LSM to each element and assuming that arbitrary nodal forces are applied to the pivot nodes, we formulate the equilibrium problem of the wall into the following variational form

$$\min_{\hat{\varphi} \in R^n} E_h(\hat{\boldsymbol{\varphi}}) = \frac{1}{2}\mathbf{P}(\hat{\boldsymbol{\varphi}}) \cdot \mathbf{A}\mathbf{P}(\hat{\boldsymbol{\varphi}}) - \mathbf{R}(\hat{\boldsymbol{\varphi}}) \cdot \bar{\mathbf{u}} \tag{4}$$

$$\text{such that}: \begin{cases} S\hat{\varphi} - \mathbf{q} = 0 \\ P(\hat{\varphi}) \leq 0 \quad \text{in ENT elements} \end{cases}$$

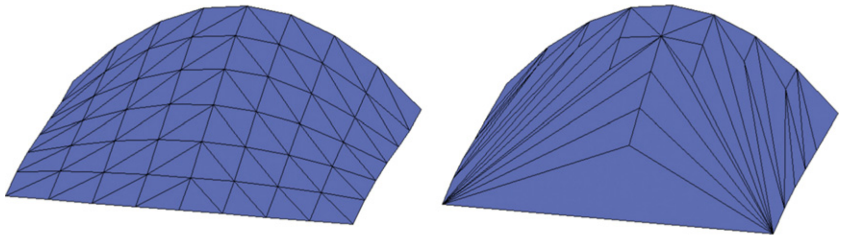

Fig. 2. (Left) Elastic solution in terms of the Airy stress function for a clamped masonry beam. (Right) Corresponding concave hull.

where $\hat{\boldsymbol{\varphi}}$ is the vector collecting the nodal values of the Airy stress functions of each element; $\mathbf{P}(\hat{\boldsymbol{\varphi}})$ *is the vector collecting the lumped stresses* $P_n^s = [[\partial\hat{\varphi}/\partial h]]_n^s$; $\mathbf{A}$ is the compliance matrix defined through (8); $\mathbf{R}(\hat{\boldsymbol{\varphi}})$ is the vector collecting the support reactions; $\bar{\mathbf{u}}$ is the vector of the imposed nodal displacements; $\mathbf{q}$ is the vector collecting the nodal forces applied to the pivot nodes; $\boldsymbol{S}$ *is the coefficient matrix of the equilibrium equations of the pivot nodes*.

In order to solve (4), we introduce the augmented Lagrangian given by

$$L_A(\hat{\boldsymbol{\varphi}}, \mathbf{u}, \rho) = E_h(\hat{\boldsymbol{\varphi}}) - \mathbf{u} \cdot (S\hat{\boldsymbol{\varphi}} - \mathbf{q}) + \frac{\rho}{2}(S\hat{\boldsymbol{\varphi}} - \mathbf{q})^2 \tag{5}$$

where $\mathbf{u}$ denotes the vector collecting the displacements of the pivot nodes (*Lagrange multipliers*), and ρ is a penalty parameter (Nocedal and Wright, 2006). We remark that the LSDM admits nonzero nodal forces only in correspondence with pivot nodes (active nodal forces), and kinematically restrained nodes (support reactions).

An iterative solution method for the minimum problem of (5) is as follows

a) given a tentative solution $(\hat{\boldsymbol{\varphi}}^k, \mathbf{u}^k, \rho^k)$, compute $\hat{\boldsymbol{\varphi}}^{k+1}$ through the quadratic programming problem:

$$\min_{\hat{\varphi} \in R^n} L_A(\hat{\boldsymbol{\varphi}}, \mathbf{u}^k, \rho^k) \quad \text{such that} \quad \mathbf{P}(\hat{\boldsymbol{\varphi}}) \le \mathbf{0} \quad \text{in ENT elements;} \tag{6}$$

b) update the Lagrange multipliers through

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \rho^k(\boldsymbol{S}\hat{\boldsymbol{\varphi}}^{k+1} - \mathbf{q}); \tag{7}$$

c) update the penalty parameter, by $\rho^{k+1} > \rho^k$, if the norm of the residual vector $\text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ increases with respect to the previous step;
d) return to point a) with $\hat{\boldsymbol{\varphi}}^k \leftarrow \hat{\boldsymbol{\varphi}}^{k+1}$; $\mathbf{u}^k \leftarrow \mathbf{u}^{k+1}$; $\rho^k \leftarrow \rho^{k+1}$, until the norm of the residual vector gets lower than a given tolerance.

A critical point is the solution of problem (6), which could be affected by lack of feasible solutions, if one does not use appropriate triangulations of the ENT members (Fraternali, 2007). We start by considering an initial triangulation of the wall and elastic solutions $\hat{\boldsymbol{\varphi}}^0$ in each element, ignoring no-tension constraints. We then determine the convex-hull $\text{conv}(\hat{\boldsymbol{\varphi}}^0)$ of $\hat{\boldsymbol{\varphi}}^0$ in the ENT members (Avis and Fukuda, 1992), and set $\hat{\boldsymbol{\varphi}}^1 = \text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ in such elements, where $\text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ is the concave face of $\text{conv}(\hat{\boldsymbol{\varphi}}^0)$. We to refer to $\text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ as the *concave hull* of $\hat{\boldsymbol{\varphi}}^0$. It is worth noting that the projection of $\text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ onto the platform defines a new triangulation of the current ENT element, which is associated with a suitable, statically admissible, stress function (cf. Angelillo and Rosso, 1995). The concave hull technique has already been used in Fraternali (2010) within a purely static context, to determine no-tension thrust networks of masonry vaults. The concave-hull driven remeshing of the ENT members may remove nodes from such elements and, eventually, create gaps between them and the neighbor elements, as we shall see in the next section (example 2). In the elastic elements we simply assume $\hat{\boldsymbol{\varphi}}^1 = \hat{\boldsymbol{\varphi}}^0$, without remeshing.

We close the present section by pointing out a key feature of the LSDM, which we plan to explore in more detail in future studies. We observe that such a method could be easily coupled with standard finite element and boundary element models. As a matter of fact, the LSDM update (7) of the pivot displacements can be rewritten as

$$\Delta\mathbf{u}^k = \mathbf{u}^{k+1} - \mathbf{u}^k = (\mathbf{K}^k)^{-1}\Delta\mathbf{q}^k \tag{8}$$

where

$$\mathbf{K}^k = \text{diag}\left\{\frac{1}{\rho^k}, \ldots, \frac{1}{\rho^k}\right\}; \qquad \Delta\mathbf{q}^k = \mathbf{q} - \mathbf{S}\hat{\mathbf{j}}^k \tag{9}$$

The association of LSDM elements with finite elements and/or boundary elements simply requires the assembly of the diagonal matrix $\mathbf{K}^k$ into the global stiffness matrix of the overall discrete model, and the insertion of the load vector $\Delta\mathbf{q}^k$ into the incremental load vector, at each update of the nodal displacements. The update of the stress function vector will be performed locally in the LSDM elements, via step a). It is worth observing that the LSDM can be easily applied to 3D structural models made up of walls that react in their own planes, and offer zero reaction to out-of-plane forces. This is a commonly accepted assumption for most masonry structures, and especially for historical buildings, due to the very low tensile strength of aged masonry (cf., e.g., Heyman, 1995).

4. Numerical results

We present in this section some numerical applications of the LSDM procedure, which refer to recurrent elements of masonry structures. The first two examples deal with a masonry beam clamped at the ends and subject to uniform vertical loading on the top edge. The two examples differ each other because in the first one there is no reinforcement of the beam, while in the second one the beam is strengthened with a steel profile at the bottom edge. The same examples were analyzed in Fraternali (2007) by means of a pure LSM approach (no displacement computation). We assume an orthotropic ENT constitutive model for the masonry characterized by the following engineering moduli: $E_1 = 6.07$ MPa, $E_2 = 5.08$ MPa, $G_{12} = 2.73$ MPa, $\nu_{12} = 0.22$. Regarding the steel reinforcement (second example), we instead assume an isotropic elastic behavior with Young modulus $E = 206,000$ MPa and Poisson ratio $\nu = 0.30$. We show the elastic stress function $\hat{\boldsymbol{\varphi}}^0$ obtained for the first example in Fig. 2, left (absence of no-tension constraints) and the corresponding set $\text{conv}(\hat{\boldsymbol{\varphi}}^0)^+$ in Fig. 2, right. The final LSDM solution obtained for this example is illustrated in Fig. 3. One immediately recognizes it predicts an "arch-type" load resisting mechanism within the beam.

The LSDM solution for the second example is illustrated in Fig. 4. In the present case, the lumped stress network exhibited by the beam partially interests the lower edge (differently from the first example, cf. Fig. 3, left), and migrates into the reinforcing steel

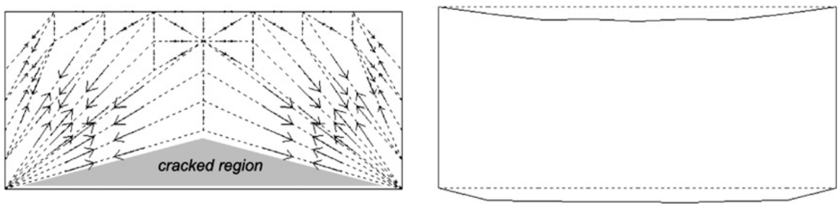

Fig. 3. LSDM solution for a clamped masonry beam. (Left) Lumped stress network. (Right) Deformed shape.

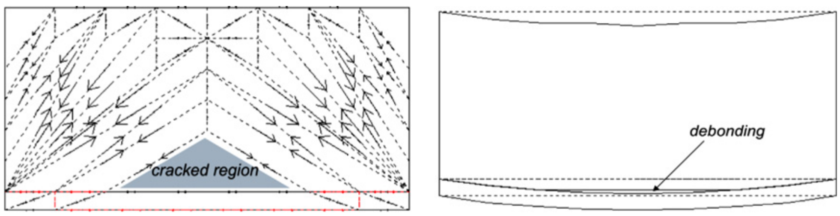

Fig. 4. LSDM solution for a clamped masonry beam strengthened with a steel profile at the bottom edge. (Left) Lumped stress network. (Right) Deformed shape.

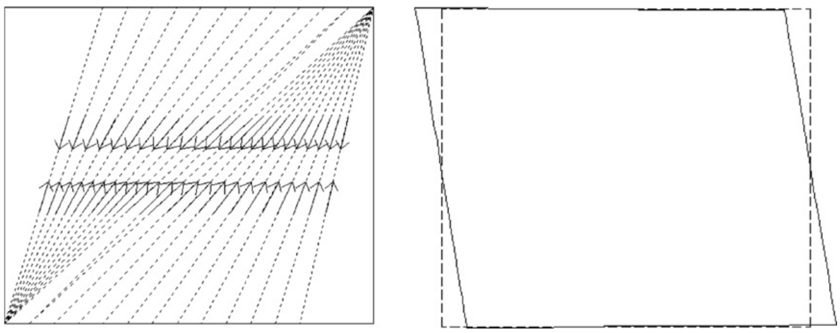

Fig. 5. LSDM solution for a shear wall. (Left) Lumped stress network. (Right) Deformed shape.

element towards the edges (Fig. 4, left). The latter is interested by alternating tensile (red) and compressive (black) stresses, as expected in a clamped beam that reacts in tension. Concerning the deformed shape, we observe that the LSDM solution predicts the debonding of the masonry beam from the steel element nearby the mid-span (Fig. 4, right). This is due to the fact that the central lower portion of the masonry beam is essentially under zero stress (Fig. 4, left), as in the previous example (Fig. 3, left). According to the ENT model, such a region is expected to be heavily cracked and not contributing to the equilibrium of the remaining portion of the body (cf. Giaquinta and Giusti, 1989; Del Piero, 1989).

The third and final example considers a shear wall subject to relative horizontal displacements of the edges. In this case, we modeled the masonry as an isotropic ENT material with zero Poisson ratio. The LSDM solution for the present problem is illustrated in Fig. 5. It is worth noting that the lumped stress pattern shown in Fig. 5,left closely reproduces the distribution of compression rays obtained in Fortunato (2000) through an analytical approach (cf. Fig. 7 of Fortunato, 2000). The ENT model predicts that the wall under consideration can be affected by cracks along such directions (Giaquinta and Giusti, 1989; Del Piero, 1989; Fortunato, 2000).

5. Concluding remarks

We have presented a mixed lumped stress–displacement method for the elastic problem of walls that include ENT elements. The LSDM generalizes the lumped stress approach to no-tension bodies presented in Fraternali (2007, 2010) and allows the computation of selected nodal displacements of the overall discrete model, in association with the nodal values of the Airy stress function. The selected displacements refer to inter-element and boundary nodes. Benchmark examples dealing with recurrent structural elements have shown that such a method is able to predict some essential features of the actual behavior of masonry walls, like, e.g., arch-type stress flow, distribution of compression rays and fracture of the material.

In closing, we point out a number of limitations of the present study that suggest directions for future work. The literature available to date includes convergence proofs for LSM-type approaches (cf. Davini, 2002; Fraternali, 2007) that do not account for unilateral (no-tension/no-compression) constraints, distorted meshes and mixed displacement–stress approaches. One obvious extension of the present numerical study therefore regards the inclusion of such effects, within a comprehensive mathematical analysis of thrust network approaches to masonry structures. We conjecture that such a study could lead to prove weak convergence of the LSM approach under general assumptions, but most of the technical questions related to the above mentioned points are still open, and their inclusion in the analysis presented in Davini (2002) and Fraternali (2007) is still far from being complete. Other challenging extensions of the present work, which we address to future work, regard the application of the LSDM to large-scale masonry structures, and the study of the link between such a procedure and standard finite element and boundary element approaches.

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