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Theoretical and Applied Fracture Mechanics 47 (2007) 274–282
theoretical and applied fracture mechanics
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Free discontinuity finite element models in two-dimensions for in-plane crack problems

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Available online 13 March 2007

Abstract

Two different free discontinuity finite element models for studying crack initiation and propagation in 2D elastic problems are presented. Minimization of an energy functional, composed of bulk and surface terms, is adopted to search for the displacement field and the crack pattern. Adaptive triangulations and embedded or *r*-adaptive discontinuities are employed. Cracks are allowed to nucleate, propagate, and branch. In order to eliminate rank-deficiency and perform local minimization, a vanishing viscosity regularization of the discrete Euler–Lagrange equations is enforced. Converge properties of the proposed models are examined using arguments of the Γ-convergence theory. Numerical results for an in-plane crack kinking problem illustrate the main operational features of the free discontinuity approach.

Keywords: Variational methods; Crack nucleation and growth; Free discontinuity problems; Discontinuous finite elements; *r*-Adaption; Non-convex minimization; Vanishing viscosity; Gamma convergence; Crack kinking

1. Introduction

The classical Griffith's criterion of crack propagation is based on the idea that a given amount of energy per unit surface G_c (fracture energy or toughness of the material) needs to be released from the body to the crack front, in order to make the crack grow. It refers to a crack loaded in mode I (Fig. 1a) and can be written in the form of the energy balance $G = G_c$, where G is equal and opposite to the to the rate at which the potential energy of the body changes with the crack area (*energy release rate*). Several variants of Griffith's criterion have been proposed for arbitrary loading (Fig. 1b). The *minimum strain energy density criterion* [1] assumes that crack grows in the direction of minimum energy density which coincides with the direction of dominant dilatation, while the *maximum energy release rate criterion* [2] supposes that the crack grows in the direction of maximum energy release rate by considering the *global* stationary values of the stored energy. Both criteria derive the direction of crack initiation $\mathbf{r}^*$ from energy considerations, the strain energy density criterion is based on minimizing the *local* energy density while the energy release rate criterion relies on using the *global* energy. The difference attributes to the accuracy of the numerical results. Local stress criteria have also been proposed and widely used in the technical literature,

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doi:10.1016/j.tafmec.2007.01.006

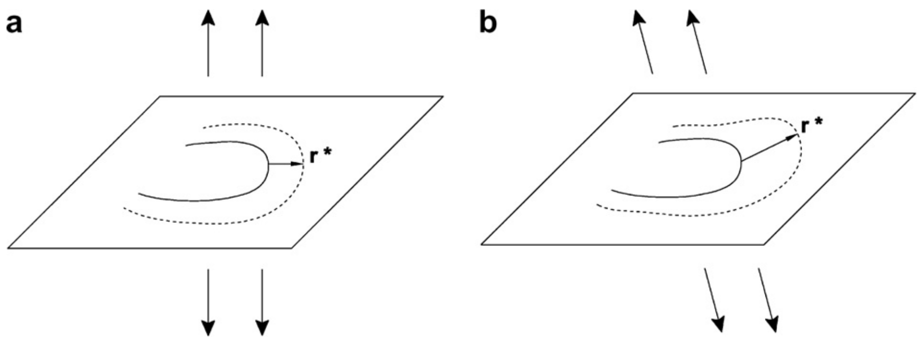

Fig. 1. Mode I (a) and mixed mode (b) propagation of an existing crack.

such as the maximum circumferential stress (or hoop stress) [3] and the maximum shear stress [4] criteria.

In recent years, further generalizations of Griffith's theory have appeared in the literature, in order to determine the crack path on the basis of an energy criterion, and also to describe crack initiation, using the framework of *Free Discontinuity Problems* [5]. Starting with [6], it has been assumed that the displacement field $\mathbf{u}$ and the crack K of a fracturing body, subject to quasistatic loading, are the arguments of an energy functional $E(\mathbf{u}, K)$ composed of bulk (potential energy) and a surface (or interface) parts. The latter represents the energy dissipated during the creation or the propagation of fractures within the body. Optimal solutions $(\mathbf{u}, K)$ are sought through a time-continuous minimization process, during which bulk and interface energies "compete" against each other, as in Griffith's criterion. Existence theorems have been proved in [7–10], for different formulations, showing that the free discontinuity approach to crack initiation and propagation is well posed, at least for brittle bodies. The original model proposed in [6] deals with a global minimization problem, but variants considering local minimization have also been proposed [11].

Concerning numerical approximations, both strong and weak approximations of the free discontinuity problem have been formulated. Strong approaches explicitly account for discontinuities and consider finite element models incorporating discontinuous test functions and mesh adaptivity [12–14]. Weak approaches instead model fracture through an auxiliary damage variable and introduce energy approximation by means of families of elliptic functionals [15]. In both cases, convergence behavior of discrete approximations has been proved [12,15], using arguments of the Γ-convergence theory [16].

The present study deals with the formulation of two different free discontinuity finite element models for 2D fracture problems in linear elasticity. Previous models considering the scalar case of antiplane shear [12] or minimization via Surface Evolver [13,14] are generalized, introducing a vanishing viscosity approach to non-convex optimization. Numerical results are given for a model problem of crack kinking under mixed loading, showing the performance and the main operational features of the two proposed approaches. Convergence properties and implementation issues are discussed, as well as challenging future directions of the present research.

2. Generalized Griffith theories

Consider the reference configuration Ω of a n-dimensional body and assume that a quasistatic boundary deformation history $\bar{\mathbf{u}}$ is prescribed on a portion $\partial_D\Omega$ of $\partial\Omega$. Suppose also that the complement $\partial_N\Omega$ of $\partial_D\Omega$ in $\partial\Omega$ is traction free and that body forces are absent (hard-device conditions).

Under such a loading history, generalized Griffith theories [6–10] model crack initiation and propagation in Ω through minimization of the energy functional

$$E(t)(\mathbf{u}, K) = \int_{\Omega\setminus K} W(\mathbf{X}, \nabla\mathbf{u}(\mathbf{X}))\mathrm{d}\mathbf{X} + \int_K \phi(\mathbf{X}, [[\mathbf{u}]](\mathbf{X}), \boldsymbol{\nu}(\mathbf{X}))\mathrm{d}H^{n-1} \qquad (1)$$

where t is the time, $\mathbf{u}$ is the displacement field of the body; $\nabla\mathbf{u}$ is the absolutely continuous part of the distributional derivative of $\mathbf{u}$; K is the crack; $\boldsymbol{\nu}$ is the unit normal to K; $[[\mathbf{u}]]$ is the difference between the values $\mathbf{u}^+$ and $\mathbf{u}^-$ of $\mathbf{u}$ on the two opposite faces of K; and H^{n-1} is the Hausdorff measure of

$(n-1)$-dimensional sets, coinciding with the ordinary length in the case of rectifiable arcs. The functions W and ϕ, appearing on the right-hand side of (1), represent bulk and interface energy densities, respectively.

For a fixed t, minimization of (1) gives rise to a *free discontinuity problem*, admitting the pair $(\mathbf{u}, K)$ as unknown, with K varying in a suitable class of hypersurfaces contained in $\bar{\Omega}$, and $\mathbf{u}$ lying in a space of sufficiently smooth functions defined over $\Omega \setminus K$. A weak formulation is obtained letting $\mathbf{u}$ be discontinuous and defined over the entire domain Ω [5]. This leads one to identify the crack K with the jump set $J(\mathbf{u})$ of $\mathbf{u}$, and to replace (1) with

$$F(t)(\mathbf{u}) = E(t)(\mathbf{u}, J(\mathbf{u})) \tag{2}$$

Existence of minimizers of (1) and (2) has been proved in spaces of special functions of bounded variation (SBV spaces), assuming that W is quasiconvex and satisfies standard coeciveness and growth conditions, the crack is a rectifiable set of bounded measure contained in Ω, and ϕ does not depend on the crack opening $[[\mathbf{u}]]$ (*brittle case*) [10]. SBV spaces collect functions whose distributional derivative is a bounded measure, composed of bulk (absolutely continuous) and interface (singular) parts (no Cantor part) [5]. The continuous-time formulation of the quasistatic evolution has been defined through a passage to the limit, introducing a time discretization $0 = t^0 < t^1 < \cdots < t^N = T$ of the loading interval $[0, T]$, and letting $N \to \infty$.

The present study deals with the special case of 2D isotropic, brittle and linear elastic bodies, assuming

$$W = \frac{1}{2}\lambda(\mathbf{X})(\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{X}))^2 + 2\mu(\mathbf{X})\boldsymbol{\varepsilon}(\mathbf{X}) \cdot \boldsymbol{\varepsilon}(\mathbf{X}) \tag{3}$$

$$\phi = G_c(\mathbf{X}) \tag{4}$$

where $\boldsymbol{\varepsilon}$ is the symmetric part of $\nabla\mathbf{u}$, while λ and μ are the Lamè coefficients corresponding to plane stress or plane strain conditions. The above choices of W and ϕ guarantee existence of solutions of the continuous-time evolution [10].

3. Free discontinuity finite element models

Let M_h denote a triangulation of Ω such that some or all of its vertices are movable, and some or all of its edges are discontinuous. Introduced a piecewise linear approximation $\mathbf{u}_h$ of $\mathbf{u}$ on M_h and denoted K_h the set of the discontinuous edges of M_h, a discrete approximation of (1) is defined through

$$E_h(\mathbf{u}_h, K_h) = \sum_{T \in M_h} \int_T W(\mathbf{X}, \nabla\mathbf{u}_h(\mathbf{X}))\mathrm{d}\mathbf{X} + \sum_{E \in K_h} \int_E \phi_h(\mathbf{X}, \delta_h(\mathbf{X}))\mathrm{d}H^1 \tag{5}$$

on a set of finite element spaces depending on the configuration of M_h. In (5), ϕ_h is the following cohesive regularization of Griffith's fracture energy (cf. [12])

$$\phi_h = G_c(\mathbf{X})\left(1 - \left(1 + b_h\frac{\delta_h(\mathbf{X})}{\delta_c}\right)\mathrm{e}^{-b_h\delta_h(\mathbf{X})/\delta_c}\right) \tag{6}$$

with δ_c a characteristic opening displacement, b_h a mesh size-dependent scale parameter, and δ_h coincident with $\|[[\mathbf{u}_h(\mathbf{X})]]\|$, $\|\cdot\|$ denoting the Euclidean norm. It is immediately verified that such a function converges to $G_c(\mathbf{X})$ for $b_h \to \infty$ (Fig. 2).

Observe that it is possible to write $\mathbf{u}_h = \mathbf{u}_h(\mathbf{X}_h^n, \mathbf{v}_h^n, \mathbf{X})$ and $K_h = K_h(\mathbf{X}_h^n)$, where $\mathbf{X}_h^n$ is the array collecting the parameters which assign the positions of the movable vertices, and $\mathbf{v}_h^n$ is the array of nodal displacements.

3.1. Solution strategy

Refer to the step $t^{n-1} \to t^n$ of a time discretization, denote all the quantities relative to $t = t^n$ by the apex n, and drop the index h for convenience. A finite element approximation $(\mathbf{u}^n, K^n)$ of the current state $(\mathbf{u}, K)$ of the body is found by minimizing the incremental functional

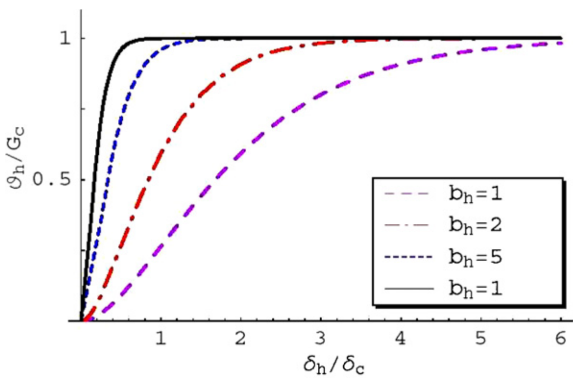

Fig. 2. Discrete cohesive regularizations of Griffith's energy.

$$E^n(\mathbf{X}^n,\mathbf{v}^n) = \sum_{T\in M^n}\int_T W(\mathbf{X},\nabla\mathbf{u}^n(\mathbf{X}^n,\mathbf{v}^n,\mathbf{X}))\mathrm{d}\mathbf{X} + \sum_{E\in K^n}\int_E \phi(\mathbf{X},\delta^n(\mathbf{X}^n,\mathbf{v}^n,\mathbf{X}))\mathrm{d}H^1 \quad (7)$$

It is worthwhile noticing that the mesh M^n needs to be adaptive, in order to expect convergence of the minimizers of (7) to those of (1). As a matter of fact, for a fixed M^n, the length of K_n cannot converge to that of the actual continuous crack K, unless the latter exactly runs along the edges of M^n, due to unrecoverable fine oscillations or "zigzag" effects [12].

On enforcing stationarity of E^n, one obtains the equations

$$\mathbf{R}^n = \frac{\partial E^n}{\partial \mathbf{X}^n} = \mathbf{0} \quad (8)$$

$$\mathbf{r}^n = \frac{\partial E^n}{\partial \mathbf{v}^n} = \mathbf{0} \quad (9)$$

requiring that residual configurational and mechanical forces be zero, respectively.

Boundary conditions for $\mathbf{v}^n$ are enforced in standard way if the imposed displacement history $\bar{\mathbf{u}}$ is uniform on $\partial_D\Omega$, as in the examples presented in the next section. In the case of non-uniform boundary conditions, Eq. (8) needs instead to be modified on $\partial_D\Omega$, as indicated in [17].

Systems (8) and (9) is generally non-convex, badly conditioned and affected by degeneracy of the Hessian matrix of E^n, even in absence of crack propagation [18]. In order to eliminate rank-deficiency, a vanishing viscosity approach can be usefully employed [11,18]. The approach used in this study generalizes that presented in [18], considering a fictitious motion, which originates from a known point $(\mathbf{X}^n_0,\mathbf{v}^n_0)$ and is ruled by the ρ,ψ gradient flow

$$\rho\frac{\mathrm{d}\mathbf{X}^n}{\mathrm{d}\tau} + \mathbf{R}^n = \mathbf{0} \quad (10)$$

$$\psi\frac{\mathrm{d}\mathbf{v}^n}{\mathrm{d}\tau} + \mathbf{r}^n = \mathbf{0} \quad (11)$$

τ denoting the fictitious time. Introduced a time discretization with time step δ, incremental updates $(\mathbf{X}^n_i,\mathbf{v}^n_i)(i=1,2,\ldots)$ are found by minimizing the sequence of functionals

$$E^n_i(\mathbf{X}^n_i,\mathbf{v}^n_i) = E^n(\mathbf{X}^n_i,\mathbf{v}^n_i) + \frac{\rho}{2\delta}\|\mathbf{X}^n_i - \mathbf{X}^n_{i-1}\|^2 + \frac{\psi}{2\delta}\|\mathbf{v}^n_i - \mathbf{v}^n_{i-1}\|^2 \quad (12)$$

whose stationarity conditions are

$$\frac{\rho}{\delta}(\mathbf{X}^n_i - \mathbf{X}^n_{i-1}) + \mathbf{R}^n_i = \mathbf{0} \quad (13)$$

$$\frac{\psi}{\delta}(\mathbf{v}^n_i - \mathbf{v}^n_{i-1}) + \mathbf{r}^n_i = \mathbf{0} \quad (14)$$

with $\mathbf{R}^n_i = \partial E^n(\mathbf{X}^n_i,\mathbf{v}^n_i)/\partial\mathbf{X}^n_i$, and $\mathbf{r}^n_i = \partial E^n(\mathbf{X}^n_i,\mathbf{v}^n_i)/\partial\mathbf{v}^n_i$.

Systems (13) and (14) is solved iteratively, through a Newton iteration with directional control [18], until $\|\mathbf{X}^n_i - \mathbf{X}^n_{i-1}\|$ and $\|\mathbf{v}^n_i - \mathbf{v}^n_{i-1}\|$ are found to be less than given tolerances. This implies that the viscous relaxation vanishes at convergence and that the final solution $(\mathbf{X}^n,\mathbf{v}^n)$ of (13) and (14) represents a local minimizer of (7) [11]. The parameter δ is chosen sufficiently small to ensure positivity of the Hessian of E^n, and is step by step rescaled.

3.2. Embedded adaptive discontinuity (EAD) model

The first free discontinuity model here presented constructs the adaptive mesh M_h by subdivision of a fixed external triangulation T_h of Ω (Fig. 3a), in such a way that each $T\in T_h$ results into four subelements (Fig. 3b). The supplementary vertices $\mathbf{X}''_k(k=1,\ldots,N'')$, introduced along the edges of T_h, are allowed to move within the segments $\mathbf{X}'_i$–$\mathbf{X}'_j$ – $(i,j=1,\ldots,N')$, according to the equations

$$\mathbf{X}''_k = \xi_k\mathbf{X}'_i + (1-\xi_k)\mathbf{X}'_j, \quad a_h \leqslant \xi_k \leqslant 1-a_h \quad (15)$$

where a_h is a mesh dependent parameter, strictly greater than zero, introduced to prevent node collapse. The array $\mathbf{X}_h$ collects the N'' variables ξ_k, appearing on the right-hand side of $(15)_1$, and is subject to constraints $(15)_2$. The discontinuity set K_h groups the edges connecting movable vertices (represented by dotted lines in Fig. 3b). This corresponds to insert duplicate nodes only in correspondence with the inner vertices $\mathbf{X}''_k$.

The present embedded discontinuity model has been proposed in [12] for the scalar problem of antiplane shear in linear elasticity. In such a case, it has been shown that the above choice of K_h permits one to reproduce all the possible rectifiable cracks of Ω, in the limit $h\to 0$, and ensures Γ-convergence of the family of discrete functionals (5) to (1), provided that it results

$$\lim_{h\to 0^+} a_h = 0, \quad \lim_{h\to 0^+} h a_h b_h = +\infty \quad (16)$$

The above result also implies convergence of the minimizers of E_h (finite-element solutions) to the minimizers of E ("exact" solutions) [12].

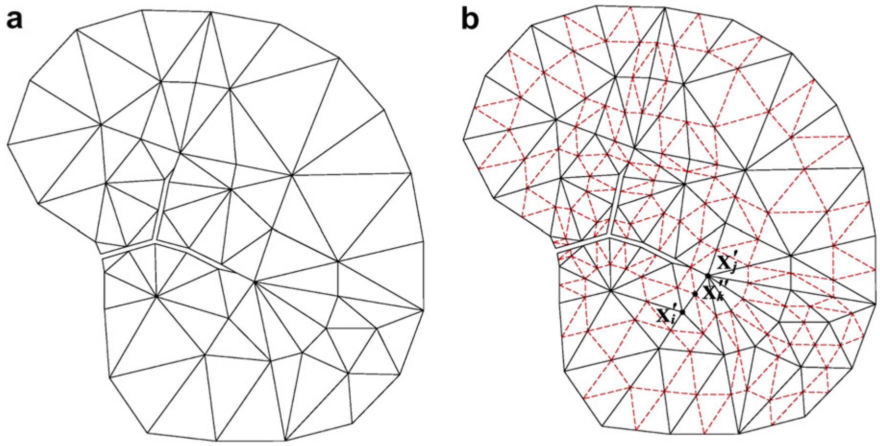

Fig. 3. Fixed triangulation (a) with inside a nested adaptive triangulation (b).

3.3. r-Adaptive discontinuity (RAD) model

A more general adaptive model is obtained by considering a triangulation M_h in which all the vertices are movable and all the edges are discontinuous. This leads to a fully *r*-adaptive discontinuous model, in which $\mathbf{X}_h$ coincides with the array of all the referential vertex coordinates, and K_h coincides with the entire set of mesh edges.

Clearly, in such a model, suitable constraints have to be imposed, in order to preserve the boundary of Ω during vertex motion. Assuming that Ω is polygonal, this implies that the vertices of Ω must remain fixed, and that edge vertices must remain within their edges [17].

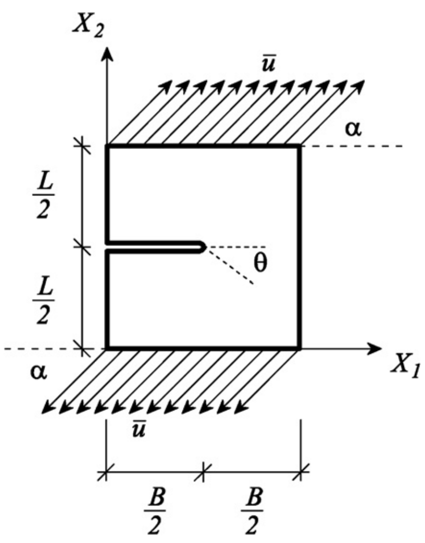

Fig. 4. Mixed mode crack kinking problem.

The generalization of the previous convergence results to the present case is rather immediate, provided that it results $\lim_{h\to 0^+} hb_h = +\infty$.

4. Numerical results

The performance of EAD and RAD approaches is tested with reference to the model problem of crack kinking under mixed mode loading, as illustrated in Fig. 4. Following [15], all the material properties are set to 1, in consistent units, as well as the aspect ratio B/L. In [15], such a problem has been analyzed through a weak free discontinuity approach, for different values of the loading angle α. Here, the attention is restricted to the case with $\alpha = 52.5°$, for which [15] estimates $\theta = 20°$ (kinking angle).

Figs. 5–7 show the results obtained through the EAD approach for three different meshes, corresponding to 4×4, 8×8 and 16×16 square grids of vertices in the external mesh T_h. The crack was allowed to start growing along an edge of T_h placed at 45° below the horizontal, and then let to run along K_h. As it is shown in Figs. 5–7 and in Fig. 12, it tends to turn and grow at 20° in correspondence with each of the employed EAD meshes, even in presence of small numerical perturbations.

The results corresponding to the RAD approach are shown in Figs. 8–11 for different mesh configurations. In detail, those of Figs. 8–10 correspond to 6×6, 8×8 and 10×10 grids of nodes, respectively, while that of Fig. 11 corresponds to a coarse mesh with a local refinement in proximity of the crack tip (RAD 24). It can be observed that mesh adaption is rather effective in the RAD approach,

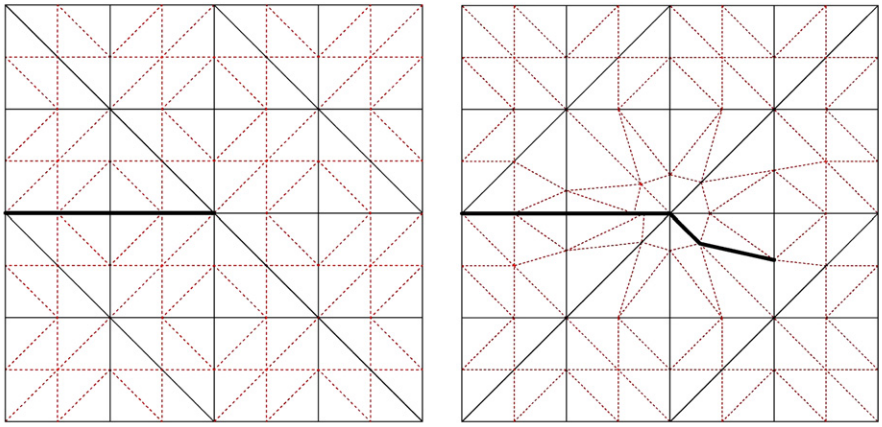

Fig. 5. Kinking prediction – EAD 4 mesh.

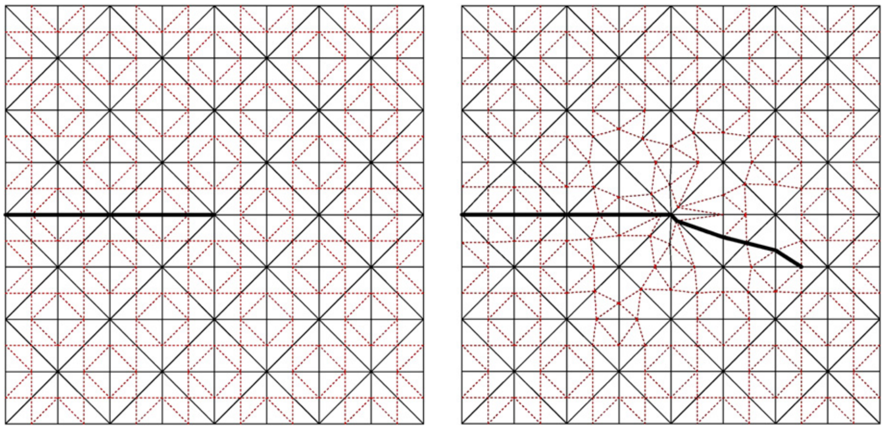

Fig. 6. Kinking prediction – EAD 8 mesh.

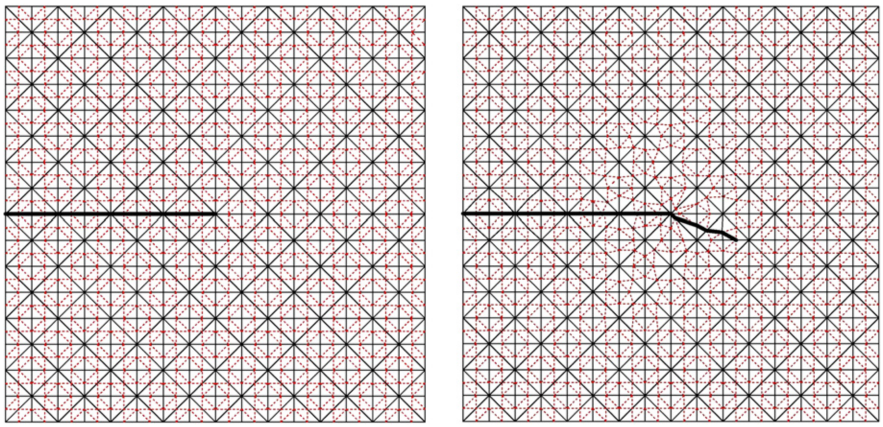

Fig. 7. Kinking prediction – EAD 16 mesh.

which is clearly more flexible than the EAD one. Also in the RAD case, the kinking angle tends to 20° below the horizontal, as the mesh size decreases (around the crack tip). It is worth noticing that all

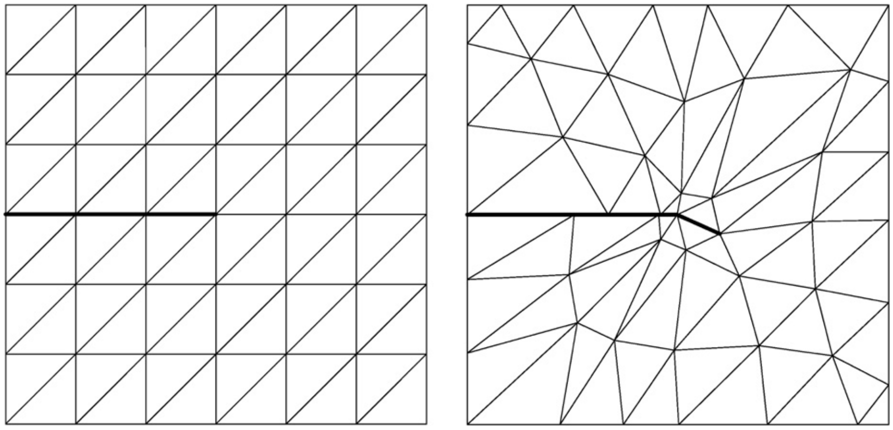

Fig. 8. Kinking prediction – RAD 6 mesh.

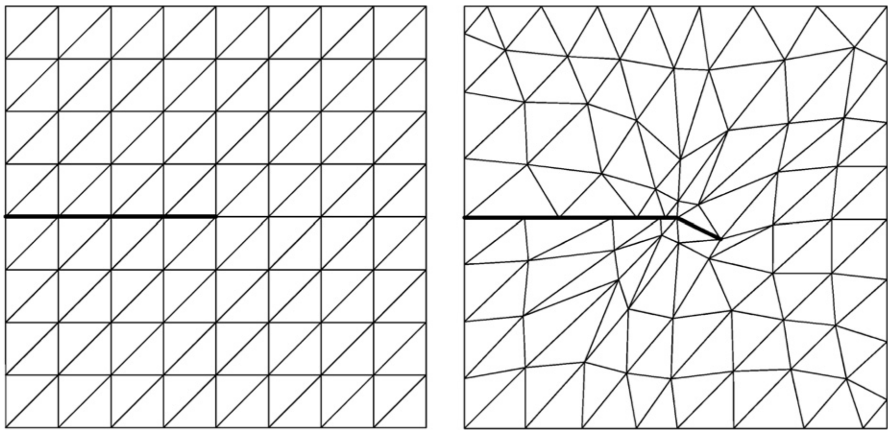

Fig. 9. Kinking prediction – RAD 8 mesh.

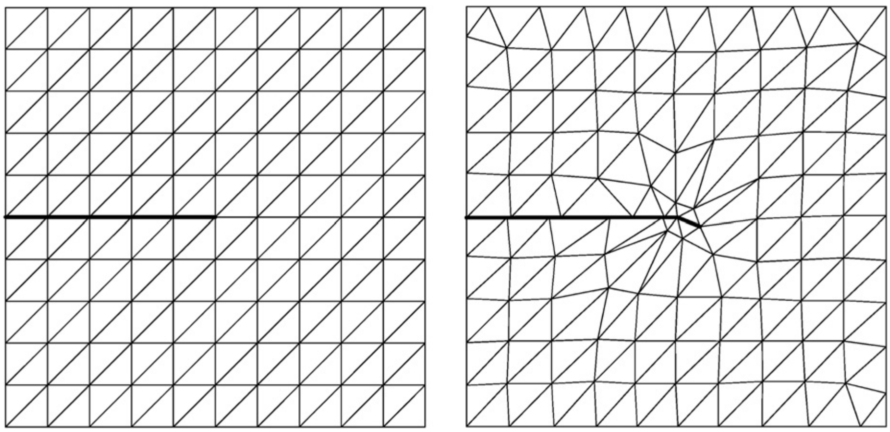

Fig. 10. Kinking prediction – RAD 10 mesh.

the examined RAD meshes initially have tip edges vertical, horizontal, or inclined above the horizontal (Figs. 8–11). They are thus not particularly tailored to the problem on hand (kinking below the horizontal), which again highlights the great flexibility of the RAD approach.

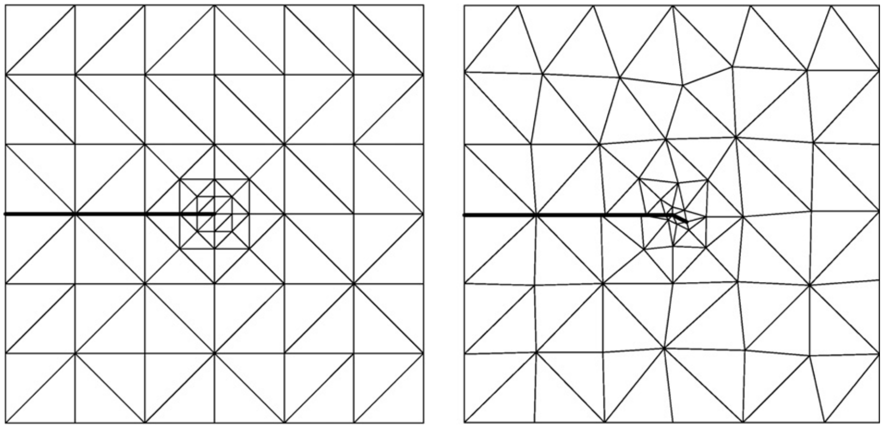

Fig. 11. Kinking prediction – RAD 24 mesh.

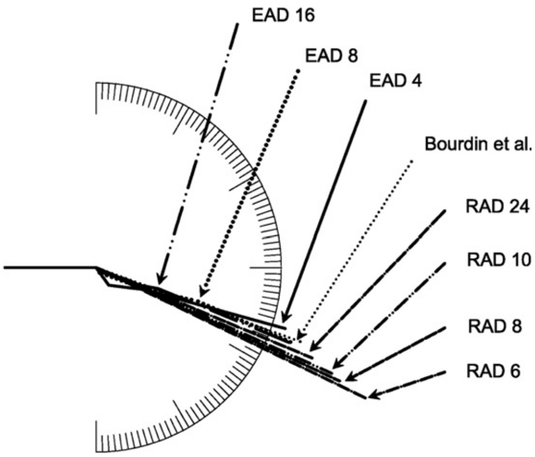

Fig. 12. Convergence of EAD and RAD predictions.

The convergence behavior of the EAD and RAD results is clearly illustrated by Fig. 12.

5. Concluding remarks and future work

Two different free discontinuity models for modeling crack initiation and propagation in brittle bodies have been presented, involving adaptive discontinuous meshes, smooth interface energies and a vanishing viscosity relaxation of the quasistatic evolution problem. Convergence of the discrete approximation has been illustrated through numerical applications and extension of previous theoretical results concerning the antiplane case of linear elasticity [12].

The presented numerical results show that the RAD approach, when associated with an efficient minimization strategy, is highly flexible and effective. Nevertheless, it requires a high computational cost, involving a large number of duplicate nodes and interface elements. The EAD approach instead implies a more limited number of degrees of freedom, but can exhibit small oscillations or zigzag effects in predicting the crack pattern, which nevertheless are going to disappear in the limit for the mesh size approaching zero. The given results also point out that excessive element distortion may cause obstruction to crack propagation, especially after the first initial steps of the evolution. Such drawback can be adjusted using remeshing, h-adaption and or topological optimization of the mesh at the beginning of each step [17,18]. Above all, the presented models appear well established from the mathematical point of view, competitive against other discontinuous approaches available in the literature, and susceptible of remarkable improvements and technical application.

Significant results are expected by future theoretical and numerical research in the field, addressing 3D fracture problems, inclusion of cohesive interface laws, local minimization and remeshing, finite elasticity or elasto-plasticity, mixed strong–weak approaches (damage to fracture transition), separation of dilatational and distorsional effects, and multi-scale modeling [19–26].

Acknowledgements

The support of the Italian MIUR through the Cofin2005 program ‘‘Modeling and Approximation Techniques in Advanced Problems of Structure and Continuum Mechanics’’ is greatly acknowledged.

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