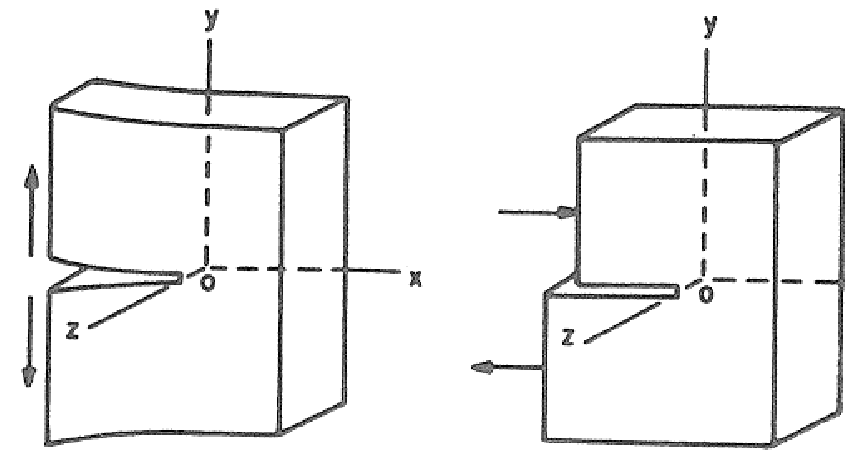
Propagazione di Fratture in Modo Misto secondo PLS

Matteo Negri

Dipartimento di Matematica - Università di Pavia

Part I. mechanics: experimental and theoretical

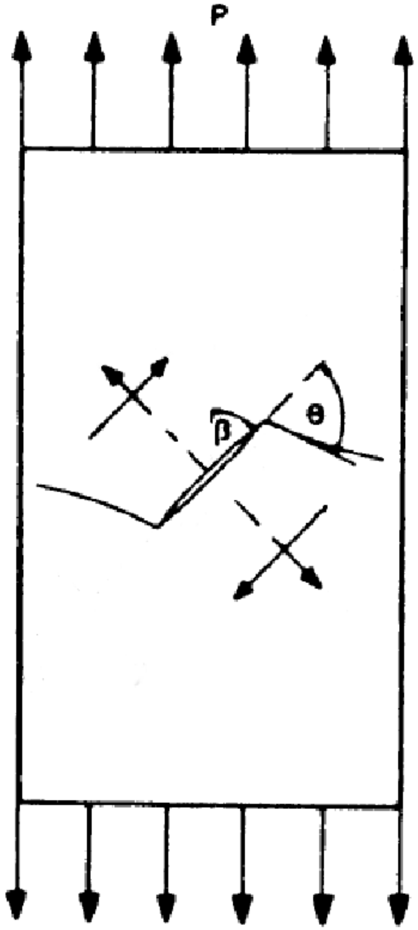
Part II. mathematics: a (regularized) model

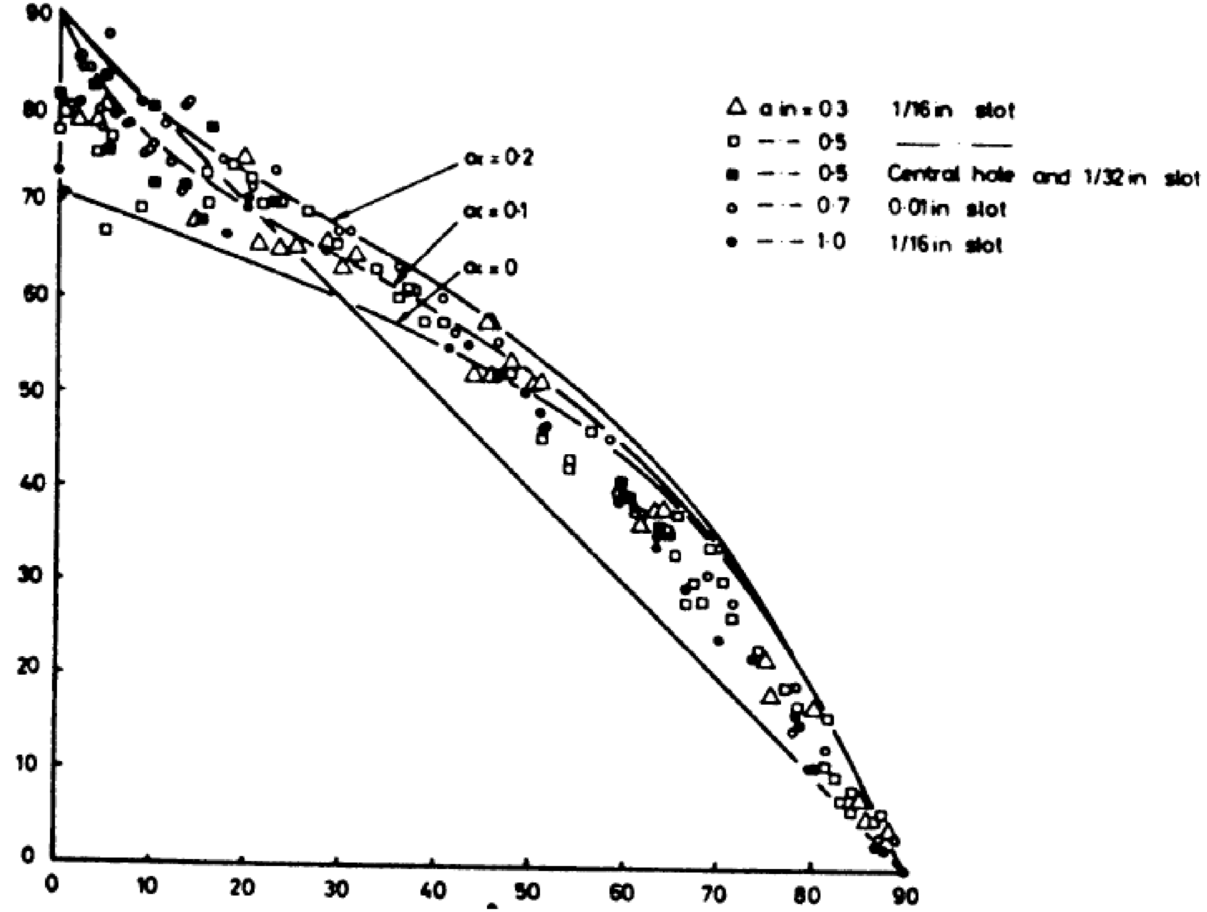


(a) OPENING (b) SLIDING

`http://www-dimat.unipv.it/~negri/`

Specimen geometry: Double Edge Notch Tension





[Williams & Ewing (71)]

A Babel of criteria

Several criteria and variations:

- principle of local symmetry (PLS) [Goldstein & Salganik (74)]
- maximum energy release rate [Cotterell (65)]
- maximum circumferential (hoop) stress [Erdogan & Sih (63)]
- strain energy density [Sih (73)]
- vectorial J-integral [Friedman & Liu (96)]
- Eshelby tensor [Kienzler & Herrman (02)]

Coincide in special cases and are just slightly different.

Stress Intensity Factors

In a (small) neighborhood of the crack tip

(in the local system of polar coordinates)

$$\boxed{\boldsymbol{\sigma} = K_I \rho^{-1/2} \boldsymbol{S}_I(\theta) + K_{II} \rho^{-1/2} \boldsymbol{S}_{II}(\theta) + \bar{\boldsymbol{\sigma}}}$$

[Irwin (51)]

$$\boldsymbol{S}_I(\theta) = (2\pi)^{-1/2} \cos(\theta/2) \begin{pmatrix} 1 - \sin(\theta/2)\sin(3\theta/2) & \sin(\theta/2)\cos(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) & 1 + \sin(\theta/2)\sin(3\theta/2) \end{pmatrix}$$

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In particular, in the DENT geometry (large domain)

$$K_I \approx p \sin^2(\beta)(\pi a)^{1/2} \qquad K_{II} \approx p \sin(\beta)\cos(\beta)(\pi a)^{1/2}$$

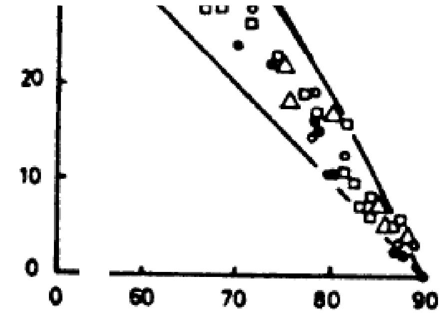
[Sih (62)]

Principle of Local Symmetry

By $K_I = p\sin^2(\beta)(\pi a)^{1/2}$ and $K_{II} = p\sin(\beta)\cos(\beta)(\pi a)^{1/2}$

$$K_{II}/K_I = \cot\beta \qquad (K_I, K_{II}) \mapsto \beta \mapsto \vartheta$$

$$K_{II} = 0 \quad \Leftrightarrow \quad \vartheta = 0 \quad \text{(no kink)}$$



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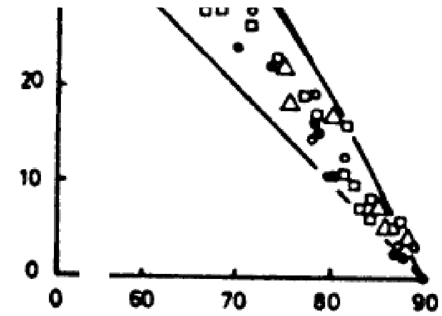

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$$\boxed{K_{II}(\Gamma_s) = 0 \ \text{ for } \ s > 0}$$

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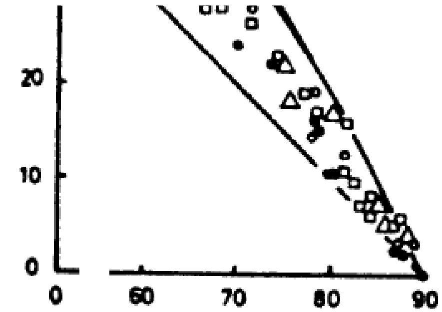

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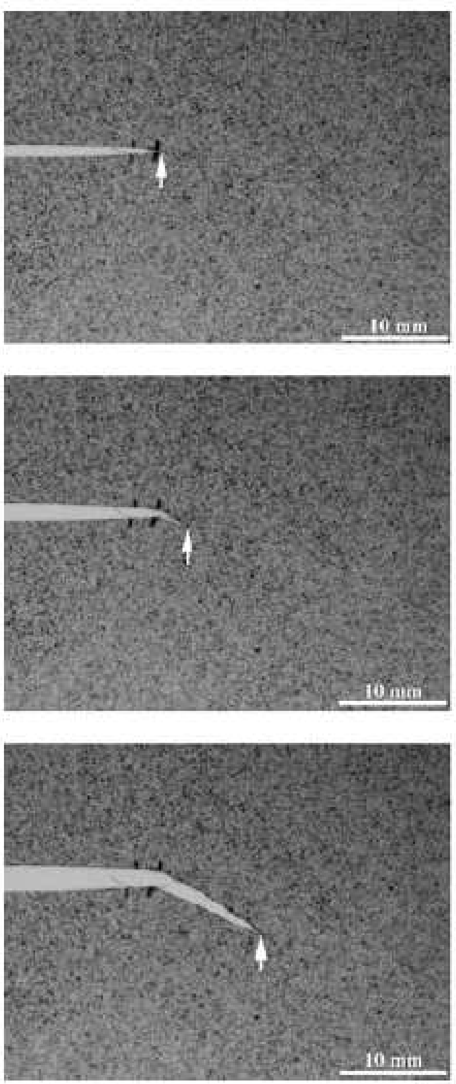
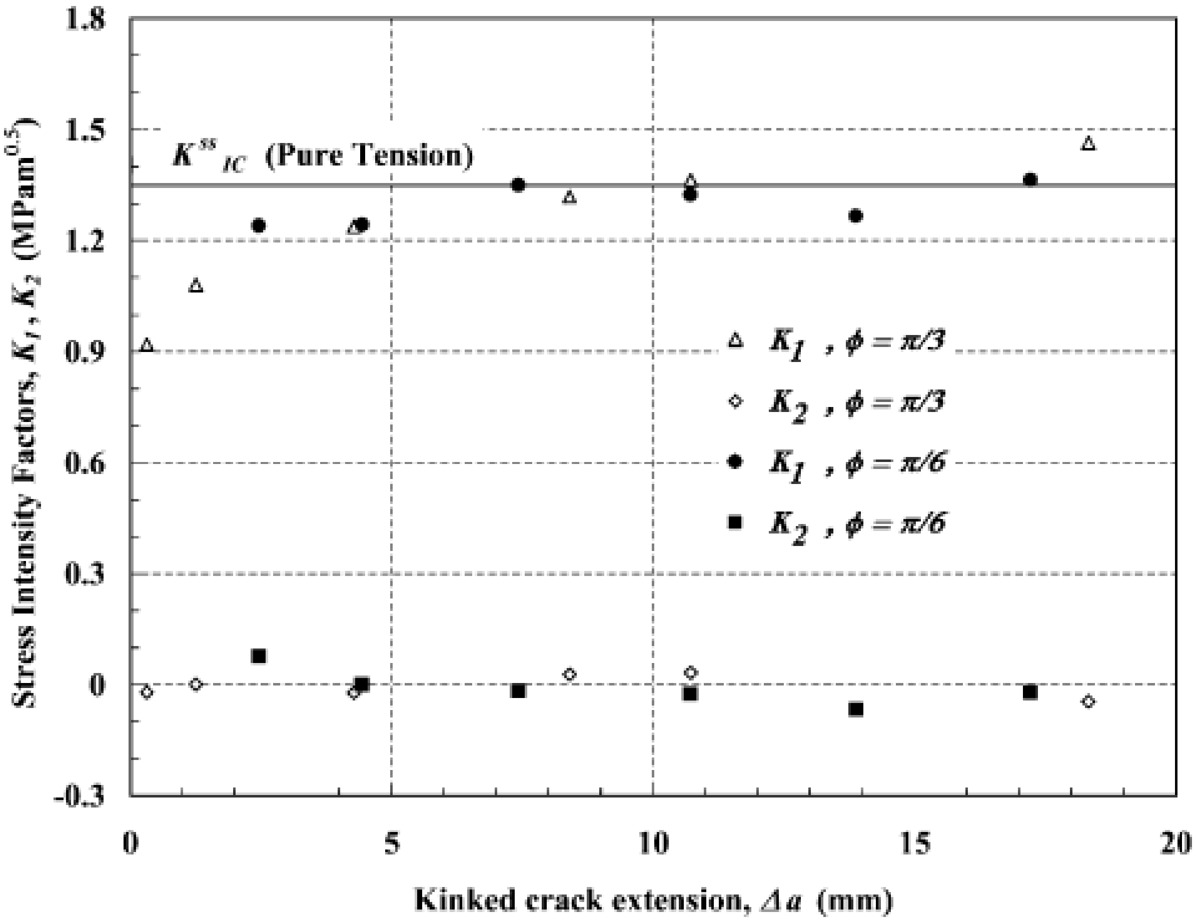
"PLS = deflection law + regularity of the crack path"

<u>At initiation</u> in general $K_{II}(\Gamma_0) \neq 0$ but $\lim_{s\to 0^+} K_{II}(\Gamma_s) = 0$

Experimental validation





[Abanto-Bueno & Lambros (06)]

Finding the kink angle θ_0

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Using the vectors $K^*(\Gamma_0, \vartheta)$ and $K(\Gamma_0)$ then

$$K^*(\Gamma_0, \vartheta) = C(\vartheta)\, K(\Gamma_0)$$

$$C(\vartheta) \approx \widetilde{C}(\vartheta) = \frac{1}{4} \begin{pmatrix} 3\cos(\vartheta/2) + \cos(3\vartheta/2) & -3\sin(\vartheta/2) - 3\sin(3\vartheta/2) \\ \sin(\vartheta/2) + \sin(3\vartheta/2) & \cos(\vartheta/2) + 3\cos(3\vartheta/2) \end{pmatrix}$$

[Williams (57), Cotterell & Rice (80)]

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Hence ϑ_0 is approximated by the solution of

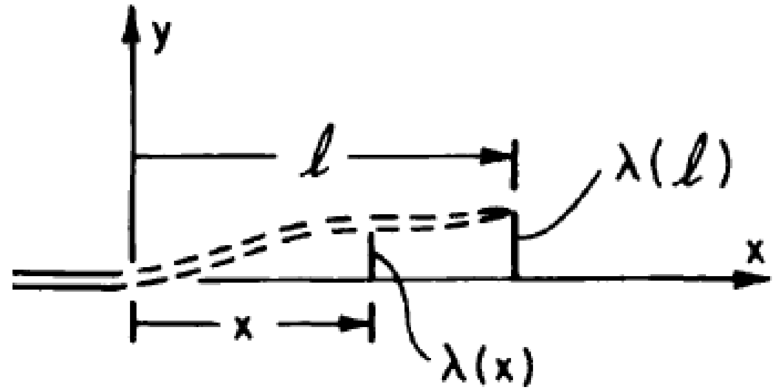
$$\widetilde{C}_{21}(\vartheta) K_I(\Gamma_0) + \widetilde{C}_{22}(\vartheta) K_{II}(\Gamma_0) = 0$$

[Under this approximation and Sih representation: PLS, hoop and G_{max} do coincide.]

Finding the crack path

Consider the classical semi-infinite crack in $\mathbb{C}$

Find a path $(x, \lambda(x))$ such that $K_{II}(\Gamma_x) = 0$ for $x > 0$.



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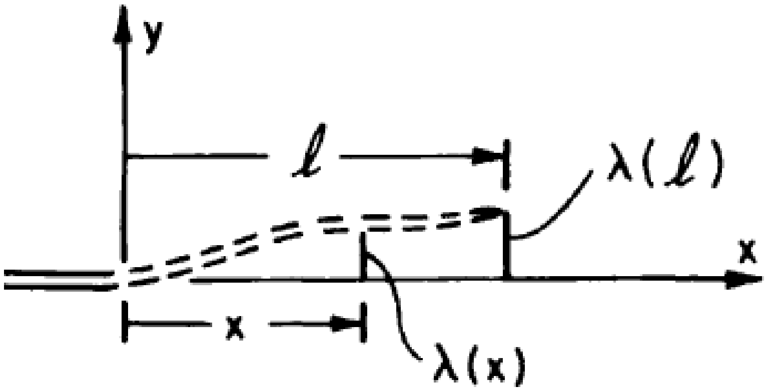

Find an expansion for $K_{II}(\Gamma_x)$ ...

[(small) perturbation of the straight path by Cotterell & Rice (80)]
[conformal mappings by Amestoy & Leblond (92)]

From $K_{II}(\Gamma_x) = 0$ ... find $\boxed{\lambda(x) \approx ax + bx^{3/2}}$

So λ is of class $C^{1,1/2}$

Finding the crack path: our mathematical setting

Consider a single edge geometry with b.c. $u = \hat{g}$ on $\partial_0\Omega$ for a Lipschitz Ω

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Requirement: given Γ

$$u_\Gamma \in \operatorname{argmin}\left\{ \int_{\Omega\setminus\Gamma} W^e(\boldsymbol{\epsilon})\,dx : u \in H^1 \text{ and } u = \hat{g} \text{ on } \partial_0\Omega \right\}$$

$$u_\Gamma = K_I \rho^{1/2}\, U_I(\theta) + K_{II}\rho^{1/2}\, U_{II}(\theta) + \bar{u}$$

(in the local system of polar coordinates)

A representation with $\bar{u} \in H^2$ is known for $\Omega \setminus \Gamma$ polygonal [Grisvard (89)]

... for y of class $C^{1,1}_{loc}$ on the base of [Lazzaroni & Toader (10)]

Approximated Stress Intensity Factors

Use an integral approximation of K_i of the form

$$\widetilde{K}_i(\Gamma_x) = \int_{\Omega\setminus\Gamma_x} (u - \mathring{u}) \cdot k_i(\theta)\, dx$$

for kernels k_i supported in $B_r(x, y(x))$ (for $r \ll 1$) of the form

[in the local system of polar coordinates]

$$k_1(\theta) = \rho^{-1/2} r^{-2} \Big(a_1 \cos(\theta/2) + a_3 \cos(3\theta/2)\, ,\; a_2 \sin(\theta/2) + a_4 \sin(3\theta/2) \Big)$$

for $\mathring{u} \approx u(x, y(x))$, e.g. $\mathring{u} = \fint_{B_{r'}} u\, dx$ for $r' \ll r$

$\widetilde{K}_i$ are well defined at least for crack paths of class $C^{0,1}$ (and $u \in H^1$)

Approximated Stress Intensity Factors (bis)

Get easily an integral approximation of $K_i^*(\Gamma, \vartheta)$ of the form

$$\widetilde{K}_i^*(\Gamma, \vartheta) = \lim_{z \to 0} \widetilde{K}_i\big(\Gamma_z\big) = \int_{\Omega \setminus \Gamma} (u - \mathring{u}) \cdot k_i(\theta - \vartheta)\, dx$$

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If $u = K_I \rho^{1/2} U_I(\theta) + K_{II} \rho^{1/2} U_{II}(\theta) + \bar{u}$ for $\bar{u} \in H^2$ then

- $|\widetilde{K}(\Gamma) - K(\Gamma)| = O(r^{1/2})$ [straight, curved cracks]
- $|\widetilde{K}^*(\Gamma,\vartheta) - \widetilde{C}(\vartheta)K(\Gamma)| = O(r^{1/2})$ [kinked cracks]

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In particular the kink angle ϑ_0 solves

$$\widetilde{K}_{II}^*(\Gamma_0,\vartheta_0) = \int_{\Omega\setminus\Gamma_0} (u-\mathring{u})\cdot k_i(\theta-\vartheta_0)\,dx = 0$$

A Functional Differential Equation for the crack path

The crack path is a graph in the set

$$\mathcal{Y} = \{y \in C^{0,1}([0,X]) \ : y(0) = 0 \text{ and } |y|_{0,1} \leq C\}$$

[choose a system of coordinates with $\hat{e}_1 = (\cos\vartheta_0, \sin\theta_0)$]

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Given $y \in \mathcal{Y}$ define an auxiliary function V

$$V : \Gamma_x \mapsto \operatorname{tg}\vartheta_x \qquad\qquad K^*_{II}(\Gamma_x, \vartheta_x) = 0$$

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$$V : \Gamma_x \mapsto \operatorname{tg}\vartheta_x \qquad\qquad K^*_{II}(\Gamma_x, \vartheta_x) = 0$$

The crack path is found by solving

$$\begin{cases} y'(x) = V(\Gamma_x) & \text{for a.e. } x > 0 \\ y(0) = 0 \end{cases}$$

[a first order FDE for the crack path]

$$y'(x) = V(\Gamma_x) = \operatorname{tg}\vartheta_x \quad \Leftrightarrow \quad \widetilde{K}^*_{II}(\Gamma_x, \vartheta_x) = \widetilde{K}_{II}(\Gamma_x) = 0$$

Ingredients of the proofs

1. existence by Schauder fixed point Theorem
2. uniqueness (still open)
3. regularity in $C^{1,1/4}([0,X]) \cap C^{1,1}_{loc}(0,X)$ (in progress)

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Properties of $\widetilde{K}_i$ (definition of V)

- behaviour for Γ_0 where u_0 has the SIF
- expansion of u_x w.r.t. x [Leblond (99), N. (11)]

$$\boxed{u_x = u_0 + x^{1/2} z_x} \qquad z_x \rightharpoonup 0 \text{ in } H^1 \text{ and } z_x \to 0 \text{ only in } H^1_{loc}$$

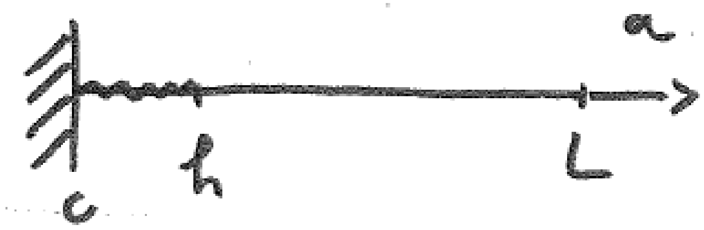
- Saint-Venant principle for Lipschitz cracks (in progress)

$$\int_{B_r} |\boldsymbol{\epsilon}(u)|^2 \, dx = 0(r)$$

"An exercise"

Consider an elastic bar $(0, L)$ with a stiffer/softer (small) inclusion in $(0, h)$

Let u_h be the equilibrium configuration with b.c. $u(0) = 0$ and $u(L) = a$.



Check that

- $\|u_h - u_0\| = O(h^{1/2})$ in $H^1(0, L)$
- $|E(u_h) - E(u_0)| = O(h)$

For $u_h = u_0 + h^{1/2} z_h$ check that

- $z_h \not\to 0$ in $H^1(0, L)$
- $z_h \rightharpoonup 0$ in $H^1(0, L)$

["The Force on an Elastic Singularity" by Eshelby (51)]

Evolution

- Consider a single edge setting for a Lipschitz Ω with <u>proportional</u> b.c.

$$u = c(t)\hat{g} \quad \text{on } \partial_0\Omega \qquad \text{with } c(0) = 0 \text{ and } c \text{ increasing}$$

- Consider plane strain linearized elasticity and brittle fracture (LEFM)
- State variables: fracture set Γ_t, displacement $u(t, x)$
- Quasi-static propagation + PLS:

  $u(t, \cdot)$ in equilibrium

  Γ_t satisfies Griffith's (equilibrium) criterion + PLS

Note that $\widetilde{K}_i(t, \Gamma) = c(t)\widetilde{K}_i(\Gamma)$ for $i = I, II$.

Fracture propagation (regularized)

Find a curve of the form $(x, y(x))$ and a parametrization $x(t)$ s.t.

- Principle of Local Symmetry

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- Griffith's Criterion (in Kuhn-Tucker fashion)

$$\widetilde{K}_I(t, \Gamma_{x(t)}) \le K_I^c \qquad \text{for } x(t) > 0 \qquad \text{[equilibrium]}$$

$$\big(\widetilde{K}_I(t, \Gamma_{x(t)}) - K_I^c\big)\, \dot{x}(t) = 0 \qquad \text{for every } x(t) > 0 \qquad \text{[flow rule]}$$

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By linearity $\widetilde{K}_{II}(t, \Gamma_{x(t)}) = c(t)\widetilde{K}_{II}(\Gamma_{x(t)}) = 0$ for $x(t) > 0$

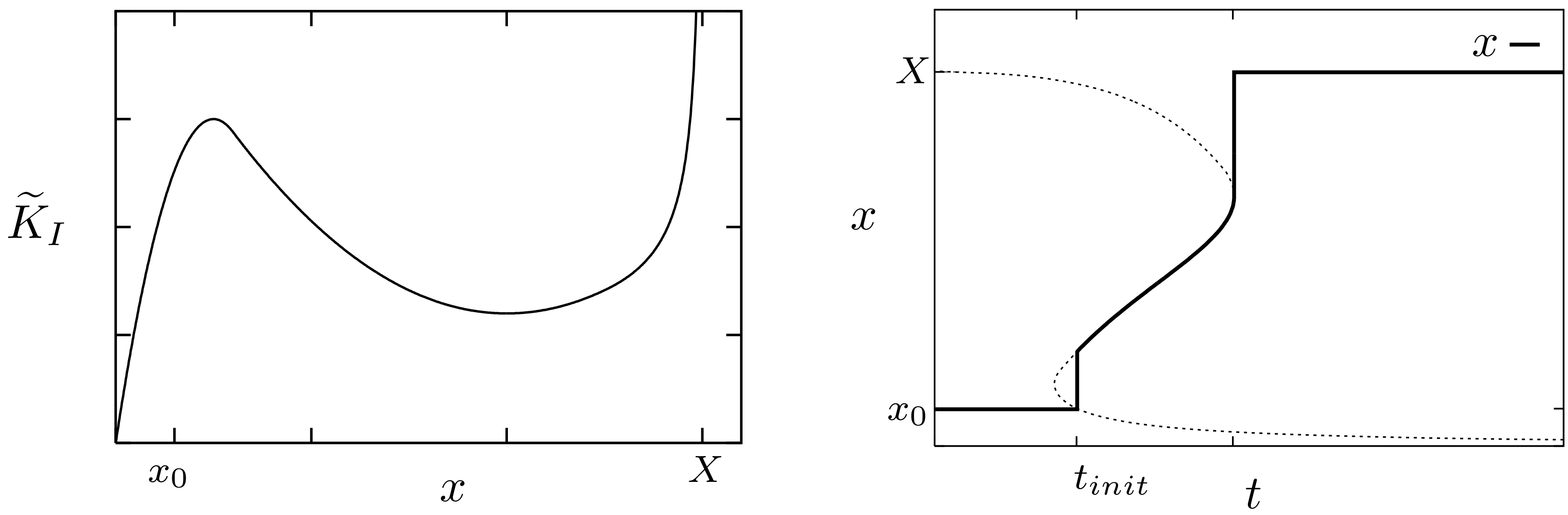
Get $\widetilde{K}_i^*(\Gamma_0, \vartheta_0) = 0$ by letting $t \searrow t_{init}$.

Parametrization: a critical example

Given the path $y \in \mathcal{Y}$, $\widetilde{K}_I(\Gamma_x)$ is continuous w.r.t. x but <u>non-monotone</u>

Horizontal crack (mode I), proportional b.c. $c(t) = ct$

($K_{II} = 0$ for the straight path)



Right: parametrization x and locus of stationary points $\{\widetilde{K}_I(t, \Gamma_x) = K_I^c\}$

Parametrization: a rate-independent pb.

Given the path $y \in \mathcal{Y}$ there exists a parametrization $x(t)$ s.t.

- the Kuhn-Tucker conditions holds: for $t_{init} = \sup\{t : x(t) = 0\}$

$$\widetilde{K}_I^*(t, \Gamma_0) \leq K_I^c \quad \text{for } t \leq t_{init} \qquad\qquad \widetilde{K}_I(t, \Gamma_{x(t)}) \leq K_I^c \quad \text{for } t > t_{init}$$

$$\big(\widetilde{K}_I(t, \Gamma_{x(t)}) - K_I^c\big)\, dx(t) = 0 \qquad \text{as a measure in } (t_{init}, T)$$

- discontinuities represents unstable regimes of the evolution.

$$\widetilde{K}_I(t, \Gamma_l) \geq K_I^c \qquad \text{for } t \in J(x) \text{ and } l \in (x(t), x^+(t))$$

[N.-Ortner (08), N. (10a)]

equivalent evolution by viscosity [Toader & Zanini (09), Knees, Mielke & Zanini (08), N. (10b)]

substantially different from the evolution by global minimizers [Francfort & Marigo (98) etc.]