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On the convergence of 3D free discontinuity models in variational fracture

Fernando Fraternali · Matteo Negri · Michael Ortiz

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Abstract Free discontinuity problems arising in the variational theory for fracture mechanics are considered. A Γ-convergence proof for an r-adaptive 3D finite element discretization is given in the case of a brittle material. The optimal displacement field, crack pattern and mesh geometry are obtained through a variational procedure that encompasses both mechanical and configurational forces. Possible extensions to cohesive fracture and quasi-static evolutions are discussed.

Keywords Variational fracture · Free discontinuity models · Γ-Convergence · r-Adaption · Configurational forces

1 Introduction

Several strong discontinuity approaches to brittle and cohesive fracture have appeared in the literature over the last years, dealing with both the mathematical and numerical aspects of such phenomena.

Most of the available discontinuous finite elements models in fracture mechanics involve embedded discontinuities (Dvorkin et al. 1990; Oliver 1996a,b), discontinuous enrichment of the displacement field via partition of unity (extended FEM) (Belytschko et al. 2001; Dolbow et al. 2000) , deformation based strategies (Hansbo and Hansbo 2004; Mergheim et al. 2005), and either local or global stress criteria of crack propagation (Ortiz and Pandolfi 1999; Gasser and Holzapfel 2005, 2006; Jäger et al. 2008).

A rich mathematical literature on the subject of brittle fracture has recently been produced in the area of calculus of variation, where the attention has mainly been focused on the application of free discontinuity models to the prediction of crack initiation and propagation (Francfort and Marigo 1998; Bourdin et al. 2000; Dal Maso and Toader 2002; Dal Maso et al. 2005; Angelillo et al. 2005; Negri 2005, 2007; Bourdin 2007; Fraternali 2007; Lussardi and Negri 2007; Negri and Ortner 2008; Schmidt et al. 2009). Free discontinuity models deal with the minimization of energy functionals composed of bulk and surface terms, which admit the crack path as a primary unknown. From the numerical point of view, such models employ smeared crack techniques (Bourdin et al. 2000; Bourdin 2007; Negri 2007; Schmidt et al. 2009), non-local approaches (Lussardi and Negri 2007), or strong discontinuity strategies (Ortiz and Pandolfi 2002; Angelillo et al. 2005; Negri 2005; Fraternali 2007; Negri and Ortner 2008).

F. Fraternali (✉)
Department of Civil Engineering, University of Salerno, 84084 Fisciano, Salerno, Italy
e-mail: f.fraternali@unisa.it

M. Negri
Department of Mathematics, University of Pavia, Via Ferrata 1, 27100 Pavia, Italy
e-mail: matteo.negri@unipv.it

M. Ortiz
Division of Engineering and Applied Science, California Institute of Technology, Pasadena, CA 91125, USA
e-mail: ortiz@aero.caltech.edu

Their distinctive feature, as compared to stress based crack path tracking strategies, consists of the fully variational formulation of the fracture problem, both in terms of the displacement field and crack pattern.

It has been found that the convergence of free discontinuity procedures requires the adoption of very fine fixed meshes (smeared crack or weak approaches), and/or adaptive triangulations (strong approaches), in order to avoid mesh-dependence and "zig-zagging" of the crack predictions (Negri 1999). Convergence studies of strongly discontinuous approaches have essentially been carried out in 2D (Negri 2005; Negri and Ortner 2008), and to our knowledge there are no available studies on such a subject in 3D. Adaptive models involve both mechanical and configurational forces, which compete against each other to determine the minimal energy solution in terms of nodal displacements, mesh geometry, and crack pattern (Thoutireddy and Ortiz 2004).

The present work deals with a Γ-convergence proof of strongly discontinuous numerical approaches to brittle fracture based on 3D r-adaptive finite element models. For a given $h > 0$, such a proof considers all the triangulations of size greater or equal to h, thus covering a large variety of finite element implementations of variational fracture.

The paper is organized as follows. Section 2 is dedicated to the statement and proof of the Γ-convergence result for finite element models of brittle fracture. Section 3 focuses on the configurational forces that drive the mesh adaption in such models. Section 4 discusses possible generalizations of the given Γ-convergence proof to cohesive fracture, while Sect. 5 analysis the variational formulation of quasi-static evolution problems. The conclusive Sect. 6 summarizes the main points of the present study and evaluates its relevance in practical implementations. Basic definitions and properties of the functional spaces examined throughout the paper are given in Appendix.

2 Statement and proof of the convergence result

For the sake of simplicity we assume that the reference configuration is represented by an open, bounded polyhedral set Ω contained in $\mathbf{R}^3$. Moreover, we assume that the set of admissible displacements $\mathcal{A}$ is given by the fields u which belong to $SBD^2(\Omega, \mathbf{R}^3)$ (see Appendix 6) and such that $\|u\|_\infty \leq k$, where $1 \ll k < +\infty$ is fixed a priori. From the physical point of view this upper bound may represent a confinement of the specimen Ω while mathematically it is essential for compactness and lower semi-continuity. As a matter of fact, this hypothesis is mainly technical; in our argument k can be arbitrarily large and its value does not effect the convergence result. (From now on, whenever possible, we will write SBD^2 instead of $SBD^2(\Omega, \mathbf{R}^3)$ and similarly for other functional spaces).

In the weak framework of SBD spaces the fracture is represented in a natural way by the set J_u (see the Appendix) thus for a brittle material the free energy will be of the form

$$\int_{\Omega \setminus J_u} W^e(\varepsilon)\, dx + \gamma \mathcal{H}^2(J_u),$$

where W^e is the density of linearized elastic energy, $\varepsilon = \varepsilon(u)$ is the linearized strain tensor and γ is the fracture toughness. More precisely we will consider the energy F to be defined as

$$F(u) = \begin{cases} \displaystyle\int_{\Omega \setminus J_u} W(\varepsilon)\, dx + \gamma \mathcal{H}^2(J_u) & \text{if } u \in \mathcal{A} \\ +\infty & \text{otherwise in } L^2. \end{cases} \tag{1}$$

(The choice of L^2 in the previous definition comes from the topology of Γ-convergence and can be easily replaced by any other L^p for $1 \leq p < +\infty$).

Let $\boldsymbol{X}_h$ be the family of all the tetrahedral meshes $\boldsymbol{T}_h$ of Ω such that every tetrahedron $T_h \in \boldsymbol{T}_h$ satisfies $h < diam(T_h)$. Note that from the strictly theoretical point of view it is not necessary to assume the regularity of the triangulations $\boldsymbol{T}_h$ in the sense of Ciarlet (1978) thanks to the minimization problem defined in the sequel. Then let V_h be the finite element set containing all the displacement fields u_h defined on the meshes $\boldsymbol{T}_h$ of $\boldsymbol{X}_h$ and having possible discontinuities across the element boundaries. Note that both $\boldsymbol{X}_h$ and V_h don't have a linear structure. Clearly, every $u_h \in V_h$ belongs to the space SBD^2, thus we can define the discrete energy simply by restriction to V_h, i.e.

$$F_h(u_h) = \int_{\Omega \setminus J_{u_h}} W(\varepsilon_h)\, dx + \gamma \mathcal{H}^2(J_{u_h}) \tag{2}$$

if $u_h \in V_h$ and $\|u_h\|_\infty \leq k$, while $F_h(u_h) = +\infty$ elsewhere in L^2. In the next subsections we will prove the following approximation result.

Theorem 1 *The energies F_h converge to F, in the sense of Γ-convergence and with respect to the strong topology of L^2.*

As usual Γ-convergence (Dal Maso 1993) follows from the upper and lower inequalities:

$$F(u) \leq \liminf_{h\to 0} F_h(u_h)$$

for every $u \in L^2$ and for every sequence $u_h \to u$ in L^2, and

$$F(u) \geq \limsup_{h\to 0} F_h(u_h)$$

for every $u \in L^2$ and for a suitable sequence $u_h \to u$ in L^2.

Roughly speaking these inequalities say that for every $u \in L^2$ it is possible to find a (recovery) sequence u_h which gives convergence of the energies, i.e. $F_h(u_h) \to F(u)$, and that this choice is "optimal" since in general (if $F_h(u_h)$ converges) we have only $\lim_{h\to 0} F_h(u_h) \geq F(u)$. Approximation in the sense of Γ-convergence gives a rigorous mathematical framework for the approximation of the energy F and, even if preliminary, provides a fundamental step for future developments in the direction of time dependent fracture propagation models.

Before proving our result it is worth to remark that in order to have convergence it is fundamental to allow a directional adaptivity of the mesh. This property plays indeed a crucial role in the correct approximation of the fracture energy and guarantees the mesh independence of the discrete approximation. A "standard" approach based on unstructured meshes would work for the bulk energy but would not give the correct approximation of the geometric term $\gamma\mathcal{H}^2(J_u)$. More precisely, in order to approximate the measure of a surface, let's say J_u, be means of a polyhedral surface, let's say J_{u_h}, it is not sufficient to have convergence in the Hausdorff distance which holds true for every family $\{\boldsymbol{T}_h\}$. Convergence of the surface measure requires a control on the "oscillations" of the approximating surfaces, or in other terms on the normal ν, which obviously depends on the orientation of the elements. Since the fracture set J_u is unknown in general a fixed mesh would not be able to catch the right directions. Note that here adaptivity does not mean refinement but just "re-orientation" of the tetrahedra. A mesh $\boldsymbol{T}_h \in \boldsymbol{X}_h$ which fits to J_u can be computed in many different ways, e.g. by means of configurational forces, as it will be shown in Sect. 3. In the next subsection it is instead shown a theoretically how to find $\boldsymbol{T}_h$, through a suitable r-adaption of a regular (background) mesh $\hat{\boldsymbol{T}}_h$.

2.1 Γ-Limsup inequality

It is well known that by the density results Chambolle (2005), Cortesani and Toader (1999) it is sufficient to find a recovery sequence u_h for displacements u such that J_u is a finite union of disjoint, closed 2-simplexes and such that u belongs to the Sobolev space $W^{2,\infty}(\Omega \setminus J_u)$.

The crucial point in the proof is the approximation of the jump set J_u. We will say that a set $J \subset \Omega$ is representable by a mesh $\boldsymbol{T}_h \in \boldsymbol{X}_h$ if J is a union of faces of the tetrahedra $T_h \in \boldsymbol{T}_h$. Now, the first step consists in finding a regular family $\{\boldsymbol{T}_h\}$ (in the sense of Ciarlet (1978)) such that J is representable by $\boldsymbol{T}_h$ for every h. Clearly, there are many ways of providing this family; we give in the following way which is closely related to our algorithm for mesh adaption.

Let us consider a preliminary case. For $v^S \in \mathbf{Z}^3$ we denote by S the plane $(v^S)^\perp = \{v \cdot v^S = 0\}$. Let $z_i \in S \cap \mathbf{Z}^3$ (for $i = 1, 2$) be independent vectors; we denote by S' the set

$$S' = \left\{ \sum_{i=1}^{2} \lambda_i z_i \quad \text{for} \quad 0 \leq \lambda_i \leq 1 \right\}$$

and by Z the lattice $\{z = \sum \lambda_i z_i \text{ with } \lambda_i \in \mathbf{Z}\}$. By linearity, for $w \in Z$ the sets $S'(w) = (S' + w)$ are contained in S and define a disjoint covering of S.

Moreover, considering the vectors $v_i = c_i \hat{e}_i$ for $c_i \in \mathbf{Z}$ $(i = 1, \ldots, 3)$ we denote by C the cuboid

$$C = \left\{ \sum_{i=1}^{3} \lambda_i v_i \quad \text{for} \quad 0 \leq \lambda_i \leq 1 \right\}, \tag{3}$$

and we denote by $C(w)$ the set $C + w$ for $w \in \mathbf{Z}^3$.

Since our proof is based on periodicity, we show that for $v^S \in \mathbf{Z}^3$ there exist two independent vectors $z_i \in S \cap \mathbf{Z}^3$ and a cuboid C of the form (3) in such a way that $S'(w) = S \cap C(w)$ for every $w \in Z$ (see Fig. 1). We can prove this property in the following way.

If $v^S = c_i \hat{e}_i$ (for some $i = 1, \ldots, 3$) then it is obvious that $C = [0, 1)^3$ satisfies the required properties.

If $v^S = c_i \hat{e}_i + c_j \hat{e}_j$ (for $i \neq j$ and $c_i \neq 0 \neq c_j$) then (see Fig. 1) we can consider

$$z_1 = c_j \hat{e}_i - c_i \hat{e}_j \qquad z_2 = \hat{e}_k \quad \text{for} \quad i \neq k \neq j$$

and the cuboid C generated by $c_j \hat{e}_i$, $-c_i \hat{e}_j$, and $\hat{e}_k$.

For the remaining case, which is is slightly more complex, it is sufficient to consider the case $z_i \in S \cap \mathbf{Z}^3$ of the form

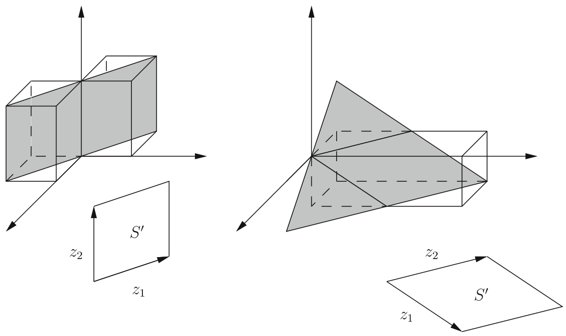

Fig. 1 Two examples of planar surfaces S and cuboids C

$$z_1 = (0, z_{1,2}, z_{1,3}) \qquad z_2 = (z_{2,1}, z_{2,2}, 0).$$

By linearity, it is not restrictive to assume that $z_{1,2}$ and $z_{2,2}$ are both positive; then the vectors

$$v_1 = z_{2,1}\hat{e}_1, \quad v_2 = (z_{1,2} + z_{2,2})\hat{e}_2, \quad v_3 = z_{1,3}\hat{e}_3$$

define a cuboid C of the form (3) with the required properties.

In the above construction, (up to a rescaling) it is not restrictive to assume that $|z_i| \gg 1$. Then, consider a (structured) tetrahedral mesh $\hat{\boldsymbol{T}}$ in $\mathbf{R}^3$ periodic with respect to the unit vectors $\hat{e}_i$ and thus with respect to vectors z_i (for $i = 1, 2$). In the case $v^S = c_i\hat{e}_i$ the surface S is easily represented by $\hat{\boldsymbol{T}}$ but in general it won't. Nonetheless for $|z_i| \gg 1$ we can find a bijection $M : C \to C$, piecewise affine on $\hat{\boldsymbol{T}}(C)$, periodic with respect to z_i (for $i = 1, 2$) and such that the tetrahedral mesh $\boldsymbol{T} = M(\hat{\boldsymbol{T}})$ represents $S \cap C$. Then, we can extend M by periodicity to the cuboids $C(w)$ (for $w \in Z$) and finally to the whole space $\mathbf{R}^3$ just setting $M(x) = x$ for all the knots $x \notin C(w)$. In this way, $\boldsymbol{T}$ is regular, since it is periodic. Hence the family $\{\boldsymbol{T}_n = 2^{-n}\boldsymbol{T}\}$ is regular as well and (by linearity) S is represented by $\boldsymbol{T}_n$ for every n.

Now, for $\zeta_i \in \mathbf{Z}^3$ $(i = 1, \ldots, 3)$ with $|\zeta_i| \gg 1$ let J be the 2-simplex

$$J = \left\{ \sum_{i=1}^{3} \lambda_i \zeta_i \quad \text{for} \quad 0 \le \lambda_i \le 1 \quad \text{and} \quad \sum_{i=1}^{3} \lambda_i = 1 \right\}.$$

It is not restrictive to consider that $\zeta_3 = 0$ (up to a change of coordinates); moreover (as $\zeta_i \in \mathbf{Z}^3$) the normal vector $v^S = \zeta_1 \times \zeta_2$ belongs to $\mathbf{Z}^3$ as well. Now, let z_i and C be respectively the vectors and the cuboid defined before for the surface $S = \{v^S \cdot v = 0\}$ and let $S' = S \cap C$. Since J is a simplex, in general it is not representable as the union of the sets $S'(w)$; nonetheless we can always cover J with a finite number of sets $S'(w_j)$ (for $j = 1, \ldots, m$). Then, in every cuboid $C(w_j)$ we can consider the mesh $\boldsymbol{T} = M(\hat{\boldsymbol{T}})$ which gives a representation of $S'(w_j)$. Finally, setting $M(x) = x$ for $x \notin C(w_j)$ we get a regular mesh which represents the union J' of the sets $S'(w_j)$ for $j = 1, \ldots, m$.

Starting from $2^{-n}\hat{\boldsymbol{T}}$ the same reasoning gives a mesh $\boldsymbol{T}_n$ which represents the union J'_n of the sets $S'(w_j)$ for $j = 1, \ldots, m_n$. For $n \to +\infty$ we have $\mathcal{H}^2(J'_n \setminus J) \to 0$. Moreover, the family $\{\boldsymbol{T}_n\}$ is again regular by self similarity.

Finally, since the definition of $\boldsymbol{T}_n$ is localized in a neighborhood of J we can repeat the same argument for finitely many disjoint simplexes.

Let us go back to the Γ-limsup inequality. Let $\{\boldsymbol{T}_n\}$ be the sequence of regular meshes constructed in the previous reasoning. Let u such that J_u is the union of finitely many disjoint simplexes, whose normal vectors belongs to a dense subset of the 2-dimensional sphere. Considering that set of vertices of $\{\boldsymbol{T}_n\}$ is $2^{-n}\mathbf{Z}^3$ it is sufficient to choose unit vectors of the form $v/|v|$ where $v \in 2^{-n}\mathbf{Z}^3$. Then, the recovery sequence u_n is defined simply as the piecewise linear Lagrange interpolation of u on $\Omega \setminus J_u$. It should be noted that, since $u \in W^{2,\infty}(\Omega \setminus J_u)$, by the Sobolev embeddings u is continuous up to the boundary, and hence uniformly continuous, and therefore the right and left continuous extensions of u on J_u exist. Then, for every vertex x of $\boldsymbol{T}_h$ we can define u_h as $u_h(x) = u(x)$ if $x \notin J_u$ and $u_h^{\pm}(x) = u^{\pm}(x)$ if $x \in J_u$. In particular, $J_{u_h} \equiv J_u$.

By standard results on finite element interpolation (cf Ciarlet (1978) Theorem 3.2.1) $u_h \to u$ in $W^{1,2}(\Omega \setminus J_u)$, whence convergence of the bulk energy is obtained. The convergence of the fracture energy is immediate since $J_{u_h} \equiv J_u$ by the definition of $\{\boldsymbol{T}_h\}$.

2.2 Γ-Liminf inequality

Since the discrete functional is defined by restriction, the Γ-liminf inequality follows trivially from lower-semicontinuity. Thus, Let $u_h \to u$ in L^2 and such that $F_h(u_h) \le C < +\infty$. Then, since u_h uniformly bounded in L^∞, it follows that u_h is bounded in BD and

$$F(u) \le \liminf_{h \to 0} F(u_h) = \liminf_{h \to 0} F_h(u_h) \tag{4}$$

by the lower semi-continuity (cf Bellettini et al. (1998) Theorem 1.3), as required.

2.3 Compactness

Let $u_h \in V_h$ such that $F_h(u_h) \leq C < +\infty$ and $\|u\|_\infty \leq C < +\infty$. Then by Bellettini et al. (1998) Theorem 1.1 we get that u_h is precompact in L^1 (indeed in L^p for every $1 \leq p < +\infty$). A remarkable consequence of compactness, which follows from a standard result in the theory of Γ-convergence (Dal Maso, 1993), is that a family of minimizers u_h^{min} of F_h converges to a minimizer u of the limit energy F.

3 Configurational forces

We have seen in the previous section that the family of meshes $\boldsymbol{X}_h = \{\boldsymbol{T}_h\}$ can be generated from a single (regular) background mesh $\hat{\boldsymbol{T}}_h$ through r-adaption (Thoutireddy and Ortiz 2004). We now want to derive the expressions of the bulk and surface configurational forces (Gurtin 2000) that permit such an adaption, in variational form.

Let $\hat{T}^e$ $(e = 1, \ldots, N_e)$ and $\hat{J}^s$ $(s = 1, \ldots, N_s)$ denote the generic tetrahedron forming $\hat{\boldsymbol{T}}_h$, and the generic triangle forming the corresponding fracture set $\hat{J}_{u_h}$, respectively. The maps $f^e : \hat{T}^e \rightarrow T^e$ and $g^s : \hat{J}^s \rightarrow J^s$ describe the transformation of $\hat{\boldsymbol{T}}_h$ into the current mesh $\boldsymbol{T}_h$, which is formed by bulk elements T^e and fracture elements J^s. We denote the total number of vertices of all the meshes in $\boldsymbol{X}_h$ by N_v, and the vertex coordinates in $\boldsymbol{T}_h$ by x_{n_i} $(n = 1, \ldots, N_v; i = 1, 2, 3)$.

The variation of the discrete energy F_h with respect to the mesh transformation is given by

$$\delta F_h = \sum_{e=1}^{N_e} \int_{\hat{T}^e} \left[\delta W \, det(\hat{\nabla} f^e) + W \, \delta(det(\hat{\nabla} f^e)) \right] d\hat{x} + \gamma \sum_{s=1}^{N_s} \delta \left(|g^s_{,1} \times g^s_{,2}| \right) \mathcal{H}^2(\hat{J}^s)$$

Expanding δF_h in terms of the quantities δx_{n_i} and collecting terms, we obtain the bulk configurational forces $R^b_{n_i}$ and the surfaces configurational forces $R^s_{n_i}$ as Thoutireddy and Ortiz (2004)

$$R^b_{n_i} = \sum_{e=1}^{N_e} \int_{\hat{T}^e} M_{ji} \, N_{n,j} \, det(\hat{\nabla} f^e) \, d\hat{x}$$

$$R^s_{n_i} = \gamma \sum_{s=1}^{N_s} \frac{\left(g^s_{,1} \times g^s_{,1}\right)_j}{\left|g^s_{,1} \times g^s_{,2}\right|} \varepsilon_{jik} \left[\left(g^s_{,2}\right)_k \hat{N}_{n,1} - \left(g^s_{,1}\right)_k \hat{N}_{n,2} \right] \mathcal{H}^2(\hat{J}^s)$$

Here, $\hat{N}_n$ is the shape function associated with the nth vertex of the background mesh $\hat{\boldsymbol{T}}_h$; N_n is the product shape function defined as $\hat{N}_n \circ (f^e)^{-1}$; ε_{jik} is the alternator symbol; and

$$M_{ij} = W \, \delta_{ij} - \left[(\nabla u_h)_{ki} \, \frac{\partial W}{\partial \, (\varepsilon_h)_{kj}} \right]$$

is the Eshelby energy-momentum tensor, δ_{ij} being the Kronecker delta. The forces $R^b_{n_i}$ act on all the vertices of $\hat{\boldsymbol{T}}_h$ and produce a mesh motion driven by elastic energy minimization. The forces $R^s_{n_i}$ instead act on the vertices of $\hat{J}_{u_h}$ and determine a crack pattern movement driven by surface energy minimization. The mechanical and configurational forces compete against each other to determine the overall equilibrium of the finite element model, in the respect of the prescribed boundary conditions (Thoutireddy and Ortiz 2004).

4 Cohesive

It is well known that brittle fracture combined with linearized elasticity is a suitable choice for few materials, like glass and ceramics (or, by the scale effect, when the dimensions of the body are very large). For other materials, like concrete and metals, it seems more realistic to consider a cohesive zone approach, where the fracture energy takes into account also the dissipative phenomena occurring in the vicinity of the crack tip (like damage for concrete and plasticity for metals).

For convenience of notation, in this section we will consider the deformation field $v = id + u$ as independent variable.

In practice, a common choice Ortiz and Pandolfi (1999) is to employ surface energy densities of the form $\phi(\delta)$; here δ is a measure of the jump $[\![u]\!] = u^+ - u^-$, while ϕ is a non-decreasing, concave function with $\phi(0) = 0$, with finite slope ϕ' for $\delta \rightarrow 0^+$ and with finite limit for $\delta \rightarrow +\infty$. A standard example is $\phi(\delta) = 1 - \exp(-\delta)$ for $\delta = ([\![u]\!]_n + \beta^2 [\![u]\!]_t)^{1/2}$, where $\beta > 0$ while $[\![u]\!]_n$ and $[\![u]\!]_t$ denote respectively the normal and tangential components of $[\![u]\!]$.

Moreover, it should be more appropriate to use non-linear elastic energies; considering for instance a Neo-Hookean law, the bulk energy density would be

$$W^e(\nabla v) = \mu|\nabla v|^2 + \gamma(\det \nabla v),$$

where now γ is usually a positive, convex function with $\gamma(d) \equiv +\infty$ for $d \leq 0$ and $\lim_{d\to 0^+} \gamma(d) = +\infty$. Since the bulk energy depends on the whole deformation gradient, the natural functional framework is the space SBV instead of SBD (see the Appendix). Thus the free energy will be of the form

$$F(v) = \int W^e(\nabla v)\, dx + \int_{S_v} \phi(\delta)\, d\mathcal{H}^2 \tag{5}$$

if $v \in SBV^2$ and $F(v) = +\infty$ otherwise in L^2. Even if brittle and cohesive functionals are similar lot of technical questions about (5) are still open and an analysis like that of Sect. 2 is still far from being complete. Nevertheless, on the bases of similar results Ambrosio and Braides (1990), Braides and Coscia (1994), we can try to "conjecture" a convergence theorem, at least in a simple case, and discuss its mechanical properties. For instance, let us consider the functional

$$F(v) = \frac{1}{2}\int_{\Omega\setminus S_v} |\nabla v|\, dx + \int_{S_v} \phi([\![v]\!])\, d\mathcal{H}^2$$

for $u \in SBV^2$ and where $\phi([\![v]\!]) = (1 - a\exp(-[\![v]\!])$ for $a > 0$. Note that there is no dependence on $\det(\nabla v)$. Considering the same finite element set V_h, let the discrete energy be defined again by restriction to V_h, i.e.

$$F_h(v_h) = \frac{1}{2}\int_{\Omega\setminus S_{v_h}} |\nabla v_h|\, dx + \int_{S_{v_h}} \phi([\![v_h]\!])\, d\mathcal{H}^2$$

for $v_h \in V_h$. Then, we can reasonably expect that F_h Γ-converges to the functional $\bar{F}$ given by

$$\bar{F}(v) = \int_{\Omega\setminus S_v} W(\nabla v)\, dx + \int_{S_v} \phi([\![v]\!])\, d\mathcal{H}^2 + a|D^c v|(\Omega)$$

for $v \in BV$. Few lines are worth to explain the meaning of $\bar{F}$ from the mathematical and mechanical point of view. It is well known Dal Maso (1993) that every Γ-limit must be a lower semi-continuous functional; since (5) does not enjoy this property and since F_h are just restrictions of F it seems natural to expect that F_h will converge to $\bar{F}$, i.e. to the lower semi-continuous envelope of F. Its integral representation is not known but similar results Ambrosio and Braides (1990), Braides and Coscia (1994) suggest that it should have exactly the form written in the previous formula. In more detail, the elastic density $W(\nabla v)$ has linear growth and is defined by

$$W(\nabla v) = \begin{cases} \frac{1}{2}|\nabla v|^2 & \text{for } |\nabla v| \leq a \\ a|\nabla v| - a^2/2 & \text{otherwise,} \end{cases}$$

where $a = \lim_{[v]\to 0^+} \phi([\![v]\!])$ represents the limit value for the elastic stress in the cohesive zone model. The fracture energies in F and $\bar{F}$ are the same but we remark once more that, for the same reasons explained in the brittle case, adaptivity of the mesh is fundamental to get the right fracture term in the limit. Finally, the term $|D^c v|$ is the variation of the Cantor part of Dv (see the Appendix). Even if the form of the elastic energy for $|\nabla v| > a$ and the presence of the Cantor term are necessary to characterize the Γ-limit, it is also possible that they are just "technical effects". Indeed, following Dal Maso and Garroni (2008), it is reasonable to expect that a minimizers v_{min} of $\bar{F}$ satisfies $|\nabla v_{min}| \leq a$ and $|D^c v_{min}| = 0$. Under these conditions we would have $F(v_{min}) = \bar{F}(v_{min})$ and thus v_{min} would be also a minimizer of F. From the mechanical point of view we could interpret the change of elastic energy and the Cantor term as the effects of a diffuse dissipation: the first reminds clearly models for perfect plasticity (Ortiz and Stainier 1999) while the second, having fractal dimension between 2 and 3, seems to be related to micro-cracking and diffuse damage (Carpinteri et al. 2003).

5 Quasi-static evolution

In this section we recall briefly some ideas related to quasi-static evolutions of brittle fractures from a variational point of view. For detailed expositions about brittle fracture we refer to Francfort and Marigo (1998), Dal Maso et al. (2005), Chambolle (2003). Let $\partial_D\Omega$ be a subset of the boundary of the domain Ω. Let $[0, T]$ a time interval and let $g : \partial_D\Omega \times [0, T] \to \mathbf{R}^3$ be a (time dependent) boundary displacement. For every integer $N > 1$ let $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform time discretization of the interval $[0, T]$ with increment $\delta t = T/N$. We assume that at time t_0 there is a pre-existing crack J_0. Then, for $n = 1, \ldots, N$, the incremental free energy will be

$$F_n(u) = \int_{\Omega\setminus(J_u\cup J_{n-1})} W^e(\varepsilon)\,dx + \gamma\mathcal{H}^2(J_u \setminus J_{n-1})$$

while the set of admissible displacements will be given by $u \in SBD^2$ such that $\|u\|_\infty \leq k$ and $u(x) = g(x, t_n)$ for $x \in \partial_D\Omega$. Note that, since healing is not physically admissible the (incremental) energy takes into account the measure of $(J_u \setminus J_{n-1})$ while the bulk energy is restricted to $\Omega\setminus(J_u \cup J_{n-1})$.

Following the ideas of Griffith (1920), Francfort and Marigo (1998) we can define a quasi-static evolution in terms of the minimizers of F_n. Roughly speaking, the idea consists in finding for every integer $N > 1$ a sequence $u_0^{(N)}, \ldots, u_N^{(N)}$ defined in terms of the minimizers of F_n. As $N \to +\infty$ the time increment $\delta t \to 0$ and (under suitable stability properties) it is then possible to find a time-continuous evolution. As a matter of fact, similar variational formulations Negri (2010) are possible, either in terms of global or local minimizers or stationary points. At the time being it is not clear which of them gives better results, in particular because it is quite rare to find explicit solutions. Clearly, form the numerical point of view, the only feasible strategy is to look for local minimizers or stationary points.

6 Concluding remarks

We have presented a Γ-convergence proof of 3D finite element models of variational fracture dealing with brittle materials, strong discontinuities and r-adaptive me-shes. The given proof shows that convergence of Griffith- type energy functionals (and minimizing sequences) calls for sufficiently flexible r-adaptive models. Theoretically, the energy minimization strategy should be able to explore all the possible meshes with size greater or equal to a given $h > 0$, for $h \to 0$. The practical implementation of such a requirement is a rather challenging task, due to the strongly non-convex nature of the coupled mechanical-configurational equilibrium problem. In practice, it can only partially be accomplished, and gradient flow or viscous relaxation strategies need to be enforced to tackle the ill-conditioning of the discrete problem (Dal Maso and Toader 2002; Fraternali 2007).

Generalization of the present convergence result to cohesive materials and quasi-static evolution problems has been discussed, reviewing recent results in these research areas, conjecturing extensions of the proof given in Sect. 2, and debating some of the main open questions in the field of variational fracture.

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Appendix

This section contains the definition and the density property of the functional spaces SBV and SBD (detailed presentations can be found for instance in Ambrosio et al. (2000) and Ambrosio et al. (1997)). According to the settings of this paper Ω will be an open, bounded, polyhedral set in $\mathbf{R}^3$.

SBV spaces

The space $SBV^p(\Omega, \mathbf{R}^3)$ is the set of fields $u \in L^1(\Omega, \mathbf{R}^3)$ such that the distributional derivative Du is a measure which can be written as

$$Du = \nabla u\,\mathcal{L}^3 + (u^+ - u^-)\nu\,\mathcal{H}^2 \llcorner S_u.$$

In the previous formula:

- ∇u denotes the density with respect to the Lebesgue measure $\mathcal{L}^3$ and it belongs to $L^p(\Omega, \mathbf{R}^{3\times 3})$
- S_u is the discontinuity set of u (see Fig. 2); it is a rectifiable set and ν is its normal vector
- $u^\pm$ are the left and right traces of u on S_u while $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure (roughly speaking a surface measure).

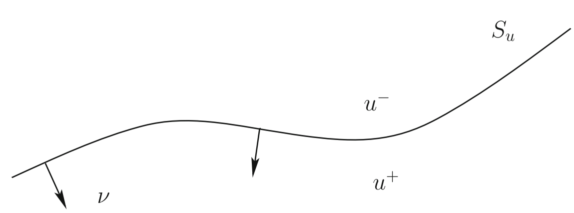

Fig. 2 A jump set S_u and some notation

SBD spaces

Employing linearized elastic energies it is more appropriate to use SBD spaces. The definition is similar to that of SBV; the space $SBD^p(\Omega, \mathbf{R}^3)$ is indeed the set of fields $u \in L^1(\Omega, \mathbf{R}^3)$ such that the symmetric part of the distributional derivative, i.e. $Eu = \frac{1}{2}(Du + Du^T)$, is a measure which can be written in the form

$$Eu = \varepsilon(u)\,\mathcal{L}^3 + (u^+ - u^-) \odot \nu\,\mathcal{H}^2 \llcorner J_u.$$

Here

- $\varepsilon(u)$ denotes the density with respect to the Lebesgue measure $\mathcal{L}^3$ and it belongs to $L^p(\Omega, \mathbf{R}^{3\times 3})$
- J_u denotes the discontinuity set of u; it is a rectifiable set and ν is its normal vector
- $u^\pm$ are the left and right traces of u on J_u

For our purposes, the following compactness and lower semi-continuity result Bellettini et al. (1998) will be fundamental.

Proposition 1 *Let u_n be a sequence in $SBD^2(\Omega, \mathbf{R}^3)$ such that*

$$\int\limits_{\Omega\setminus J_{u_n}} W\left(\varepsilon(u_n)\right)\ dx + \mathcal{H}^2\left(J_{u_n}\right) + \|u_n\|_\infty \leq C.$$

Then there exist a subsequence u_k of u_n and a function $u \in SBD^2(\Omega, \mathbf{R}^3)$ such that $u_k \to u$ in $L^1(\Omega, \mathbf{R}^3)$, $\varepsilon(u_k) \rightharpoonup \varepsilon(u)$ in $L^1(\Omega, \mathbf{R}^{3\times 3})$ and

$$\mathcal{H}^2\left(J_u\right) \leq \liminf_{k\to+\infty}\left(J_{u_k}\right).$$

Finally, we report the following density result, which follows from Chambolle (2005) and Cortesani and Toader (1999).

Proposition 2 *Let $u \in SBD^2(\Omega, \mathbf{R}^3) \cap L^\infty(\Omega, \mathbf{R}^3)$. There exists a sequence u_n such that J_{u_n} is a disjoint, finite union of 2-simplexes, $u_n \in W^{2,\infty}(\Omega \setminus J_{u_n}, \mathbf{R}^3)$ and $u_n \to u$ in $L^1(\Omega, \mathbf{R}^3)$, $\|u_n\|_\infty \leq \|u\|_\infty$, $\varepsilon(u_n) \to \varepsilon(u)$ in $L^2(\Omega, \mathbf{R}^{3\times 3})$ and $\lim_{n\to+\infty} \mathcal{H}^2(J_{u_n}) \leq \mathcal{H}^2(J_u)$.*

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