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Error Estimates for a Lumped Stress Method for Plane Elastic Problems

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The variational properties and the convergence order of a Lumped Stress Method (LSM) for 2D anisotropic elasticity are presented. Such a method can be thought of as a rational procedure to approximate a plane continuous body by a truss-like structure. The traction problem of plane elasticity is considered, making use of the Airy stress function. Under suitable assumptions, the convergence of the LSM is proved on using arguments of the mathematical theory of mixed finite element methods. The given result is useful in order to prove the accuracy of the discrete-continuum approximation in technical applications.

Keywords discrete-continuum approximations, plane elasticity, stress approaches, mixed methods, convergence analysis, error estimates, discrete force networks

1. INTRODUCTION

Mixed finite element methods are often used to approximate a given fourth-order boundary value problem with two second-order problems (*primal/dual approaches;* see, e.g., [1–10]), especially in the case of biharmonic boundary value problems (Airy's formulation of isotropic plane elasticity; bending of isotropic Kirchhoff plates; etc.).

In [1] Glowinski first proved the convergence of mixed methods for the biharmonic problem. Subsequently, Ciarlet and Raviart [2, 3] obtained the rate of convergence of mixed methods involving polynomials of degree $k \geq 2$, while Scholtz in [4, 5] deduced an analogous result for piecewise linear polynomials. In [6–8] Davini and Pitacco have proposed a Lumped Strain Method for Kirchhoff plates, obtaining convergence results through the mathematical theory of mixed methods [7], and Γ-convergence theory [8].

A result for mixed approximations of general fourth-order problems can be found in [10], where again polynomials of degree $k \geq 2$ are taken into consideration.

The present paper deals with the convergence proof of a *Lumped Stress Method* (LSM) recently appeared in the literature for anisotropic 2D elasticity [11, 12].

The LSM involves piecewise-linear approximations of the primal variable (*Airy's stress function* φ), which are defined on a given triangulation of the body (*primal mesh*). It also makes use of piecewise-constant approximations of the secondary, tensor-valued variable, which coincides with the hessian of φ. The latter is defined over a *dual mesh*.

Such a choice of approximating function spaces is based on a pre-minimization procedure inspired by the relaxation strategies discussed by Kohn et al. in [13, 14]. It leads to a complete decoupling of the dual problem from the primal one (unconstrained mixed method, cf. Davini and Pitacco [6, 7]).

The physical meaning of the LSM, numerical convergence studies and applications to relevant benchmark problems have been presented in [11, 12]. It has been shown that such a method offers the possibility to rationally approximate a continuous body through a non-conventional truss-structure [12]. The skeleton of the primal triangulation can instead represent a truss structure, whose complementary energy is defined per dual elements.

In what follows, some preliminaries about mixed approaches to fourth-order problems are given (Theorems 1, 2), and the mathematical formulation of the LSM is presented. Moreover, the convergence order of the method is obtained (Theorem 3), assuming suitable regularity assumptions about finite element meshes.

In the last section of the paper, the physical meaning of the LSM is examined and numerical applications are presented.

Differently from topology optimization methods (refer, e.g., to Bendsøe and Sigmund [15]), the stress network in the LSM is arbitrary and doesn't need to follow the principal direction of stress or other optimal directions.

Nevertheless, the association of such a method with optimal design techniques awaits attention. The use of the LSM for shape optimization of masonry structures has been presented in [16, 17].

2. VARIATIONAL FORMULATIONS OF PLANE ELASTICITY

2.1. Airy's Formulation

Let us consider the traction problem of a plane, bounded and simply connected open set Ω, owing a polygonal boundary $\partial\Omega$.

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Throughout the paper, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$ denotes a Cartesian basis, Greek indices are assumed to range over $\{1, 2\}$; summation convention over repeated indices is employed; and use is made of the *two-dimensional alternator* $e_{\alpha\beta}$.

Let $\mathbf{p}$ denote the surface traction prescribed on $\partial\Omega$; $\hat{\mathbf{n}}$ the unit outward normal to $\partial\Omega$; $\mathbf{b}$ the body force density (per unit volume); and $\bar{\mathbf{E}}$ a given field of initial strains (*eigenstrains*).

The equilibrium equations of Ω are the following

$$div\,\mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{in}\,\Omega, \tag{1}$$

$$\mathbf{T}\hat{\mathbf{n}} + \mathbf{b} = \mathbf{p} \quad \text{on}\,\partial\Omega, \tag{2}$$

where $\mathbf{T} = T_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ is the stress field.

The general solution of (1)–(2) can be expressed in terms of the Airy stress function φ (see, e.g., Gurtin [18]) as

$$\mathbf{T} = \mathbf{T}^* + \mathbf{W}^T\mathbf{H}\varphi\mathbf{W}, \tag{3}$$

where $\mathbf{T}^*$ is a particular stress field in equilibrium with $\mathbf{b}$ and $\mathbf{p}$; $\mathbf{H}\varphi$ is the *hessian* of φ

$$\mathbf{H}\varphi = \nabla(\nabla\varphi) = \varphi_{,\alpha\beta}\hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta; \tag{4}$$

and $\mathbf{W}$ is the skew tensor with components $W_{\alpha\beta} = e_{\alpha\beta}$. The function φ must be such that $\varphi(\sigma) = 0$ and $\frac{\partial\varphi}{\partial n}(\sigma) = 0$ on $\partial\Omega$, σ being the arc length along $\partial\Omega$.

We assume that the fourth order compliance tensor

$$\mathbf{A} = A_{\alpha\beta\gamma\delta}\hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta \otimes \hat{\mathbf{e}}_\gamma \otimes \hat{\mathbf{e}}_\delta, \tag{5}$$

is positive definite.

A variational formulation of the elastic problem is given by *the principle of minimum complementary energy*. It can be stated as

Find $\varphi_0 \in H_0^2(\Omega)$ *such that*

$$\mathcal{E}(\varphi_0) = \inf_{\varphi \in H_0^2(\Omega)} \mathcal{E}(\varphi). \tag{6}$$

where

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_\Omega \mathbf{H}\varphi \cdot \mathcal{A}[\mathbf{H}\varphi]da - l(\varphi). \tag{7}$$

In (7), $\boldsymbol{\mathcal{A}}$ is the *transformed compliance tensor* of components

$$\mathcal{A}_{\alpha\beta\gamma\delta} = e_{\alpha\mu}e_{\beta\nu}e_{\gamma\rho}e_{\delta\sigma}A_{\mu\nu\rho\sigma}; \tag{8}$$

$l(\varphi)$ is the linear functional

$$l(\varphi) = -\int_\Omega \mathbf{H}\varphi \cdot \mathbf{W}^T(\bar{\mathbf{E}} + \mathbf{A}[\mathbf{T}^*])\mathbf{W}da; \tag{9}$$

and

$$H_0^2(\Omega) = \left\{\varphi \in H^2(\Omega)/\varphi = 0, \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega\right\} \tag{10}$$

is the space of admissible stress functions, $H^m(\Omega)$ denoting the Hilbert space of functions which are square integrable together with their distributional derivatives up to the mth order. We refer the reader to [3, 9, 19, 20] for the mathematical background of the present study.

On applying the Green formula, it is easy to transform the linear functional (9) into the form

$$l(\varphi) = \int_\Omega f\varphi da, \tag{11}$$

where

$$f = -e_{\alpha\mu}e_{\beta\nu}\left(\bar{E}_{\alpha\beta} + A_{\alpha\beta\gamma\delta}T^*_{\gamma\delta}\right)_{,\mu\nu}. \tag{12}$$

In what follows we will use the assumption $f \in L^2(\Omega)$.

Notice that the functional (7) differs from the complementary energy of Ω by the constant term $1/2\int_\Omega \mathbf{T}^* \cdot (\bar{\mathbf{E}} + \mathbf{A}[\mathbf{T}^*])\,da$.

2.2. Mixed Formulation

Let us introduce the intermediate variable $\boldsymbol{\psi} = \psi_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, satisfying the constraint

$$\boldsymbol{\psi} = -\mathbf{H}\varphi. \tag{13}$$

A mixed formulation of (6) can be obtained on introducing the functional

$$\mathcal{F}((\varphi, \boldsymbol{\psi})) = \frac{1}{2}\int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}]\ da - \ell((\varphi, \boldsymbol{\psi})), \tag{14}$$

defined over the function space

$$\mathcal{V} = \left\{(\varphi, \boldsymbol{\psi}) \in H_0^1(\Omega) \times (L^2(\Omega))^4/\beta((\varphi, \boldsymbol{\psi}), \mathbf{q}) = 0, \; \forall\mathbf{q} \in (H^1(\Omega))^4\right\}, \tag{15}$$

where $\ell : \mathcal{V} \to \mathcal{R}$ denotes the following linear form

$$\ell((\varphi, \boldsymbol{\psi})) = l(\varphi) = \int_\Omega f\,\varphi\,da, \tag{16}$$

while $\beta : (H_0^1(\Omega) \times (L^2(\Omega))^4) \times (H^1(\Omega))^4 \to \mathcal{R}$ denotes the bilinear form

$$\beta((\varphi, \boldsymbol{\psi}), \mathbf{q}) = \int_\Omega \nabla\,\varphi \cdot div\,\mathbf{q}\,da - \int_\Omega \boldsymbol{\psi} \cdot \mathbf{q}\,da. \tag{17}$$

In the following Theorem 1, we show that the equation $\beta((\varphi, \boldsymbol{\psi}), \mathbf{q}) = 0$, $\forall\mathbf{q} \in (H^1(\Omega))^4$ represents a variational formulation of the constraint (13) and the boundary condition $\partial\varphi/\partial n = 0$ on $\partial\Omega$. We also show that problem (6) is equivalent to the following constrained minimization problem

Find $(\varphi^*, \boldsymbol{\psi}^*)$ *such that*

$$\mathcal{F}((\varphi^*, \boldsymbol{\psi}^*)) = \inf_{(\varphi,\boldsymbol{\psi})\in\mathcal{V}} \mathcal{F}((\varphi, \boldsymbol{\psi})). \tag{18}$$

The symbols $\|\varphi\|_m$ and $|\varphi|_m$ will be employed for the usual norm and seminorm of the scalar function φ in the space $H^m(\Omega)$, respectively. Moreover, the notation

$$\|\mathbf{p}\|_m = \left(\sum_{\alpha,\beta=1}^{2} \|p_{\alpha\beta}\|_m^2\right)^{1/2}, \quad |\mathbf{p}|_m = \left(\sum_{\alpha,\beta=1}^{2} |p_{\alpha\beta}|_m^2\right)^{1/2}, \tag{19}$$

will be used to denote the norm and the seminorm of the tensor-valued function $\mathbf{p} = p_{\alpha\beta}\hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta \in (H^m(\Omega))^4$.

Theorem 1. *The constrained minimization problem (19) has one and only one solution* $(\varphi^*, \boldsymbol{\psi}^*)$, φ^* *being coincident with the solution* φ_0 *of problem (6) and* $\boldsymbol{\psi}^* = -\mathbf{H}\varphi_0$.

Proof. Equipped with the product norm

$$\|(\varphi, \boldsymbol{\psi})\|_{\mathcal{V}} = \left(|\varphi|_1^2 + \|\boldsymbol{\psi}\|_0^2\right)^{1/2}, \tag{20}$$

the space $\mathcal{V}$ defined as in (15) is a Hilbert space.

Consider now the following symmetric bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathcal{R}$

$$a((\varphi, \boldsymbol{\psi}), (\varphi', \boldsymbol{\psi}')) = \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}']\, da. \tag{21}$$

Upon introducing the maximum characteristic value $\Lambda_{\max}$ of the positive definite tensor $\boldsymbol{\mathcal{A}}$ ($\Lambda_{\max} > 0$), from the definition (20) and the Cauchy-Schwartz inequality we get

$$|a((\varphi, \boldsymbol{\psi}), (\varphi', \boldsymbol{\psi}'))| \leq \Lambda_{\max} \|(\varphi, \boldsymbol{\psi})\|_{\mathcal{V}} \|(\varphi', \boldsymbol{\psi}')\|_{\mathcal{V}}, \tag{22}$$

and thus a is continuous on $\mathcal{V}$.

Furthermore, the choice $\mathbf{q} = \varphi(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2)$, i.e., $\operatorname{div} \mathbf{q} = \nabla\varphi$, in Eq. (17) gives

$$\begin{aligned}|\varphi|_1^2 = \int_\Omega \nabla\varphi \cdot \nabla\varphi\, da &= \int_\Omega (\psi_{11} + \psi_{22})\varphi\, da \\ &\leq \sqrt{2}\, \mathrm{C}(\Omega)\, \|\boldsymbol{\psi}\|_0\, |\varphi|_1, \end{aligned}\tag{23}$$

$\mathrm{C}(\Omega)$ being the Poincarè constant. The substitution of Eq. (23) into Eq. (20) leads us to write

$$\begin{aligned}\|(\varphi, \boldsymbol{\psi})\|_{\mathcal{V}}^2 &\leq (1 + 2\mathrm{C}(\Omega)^2)\, \|\boldsymbol{\psi}\|_0^2 \\ &\leq \frac{(1 + 2\mathrm{C}(\Omega)^2)}{\Lambda_{\min}}\, a((\varphi, \boldsymbol{\psi}), (\varphi, \boldsymbol{\psi})), \end{aligned}\tag{24}$$

$\Lambda_{\min}$ being the minimum characteristic value of $\boldsymbol{\mathcal{A}}$ ($\Lambda_{\min} > 0$).

Hence, a is also coercive on $\mathcal{V}$. On the other hand, since the linear form l, defined as in Eq. (16), is continuous on $\mathcal{V}$ under the assumption $f \in L^2(\Omega)$, the existence and uniqueness of the solution $(\varphi^*, \boldsymbol{\psi}^*)$ of problem (18) follow from the Lax-Milgram Lemma.

Now, observe that the following relation

$$\int_\Omega \mathbf{H}\varphi^* \cdot \mathbf{q}\, da = -\int_\Omega \nabla\varphi^* \cdot \mathit{div}\, \mathbf{q}\, da, \tag{25a}$$

holds, in the sense of distributional derivatives, for any $\mathbf{q} \in (D(\Omega))^4$, with $D(\Omega) = C_0^\infty(\Omega)$.

On the other hand, the couple $(\varphi^*, \boldsymbol{\psi}^*)$ is an element of the space $\mathcal{V}$ defined in Eq. (15), and hence it results $\beta((\varphi^*, \boldsymbol{\psi}^*), \mathbf{q}) = 0$, $\forall \mathbf{q} \in (H^1(\Omega))^4$. Since $D(\Omega) \subset H^1(\Omega)$, from Eq. (17) we deduce

$$\int_\Omega \nabla\varphi^* \cdot \mathit{div}\, \mathbf{q}\, da = \int_\Omega \boldsymbol{\psi}^* \cdot \mathbf{q}\, da, \quad \forall \mathbf{q} \in (D(\Omega))^4. \tag{25b}$$

Formulae (25a,b) imply $\mathbf{H}\varphi^* = -\boldsymbol{\psi}^* \in (L^2(\Omega))^4$. Using this results into the relation

$$\begin{aligned}&\int_\Omega (\nabla\varphi^* \cdot \mathit{div}\, \mathbf{q} - \boldsymbol{\psi}^* \cdot \mathbf{q})\, da \\ &\quad = -\int_\Omega (\mathbf{H}\varphi^* + \boldsymbol{\psi}^*) \cdot \mathbf{q}\, da + \int_{\partial\Omega} \nabla\varphi^* \otimes \mathbf{q}\hat{\mathbf{n}}\, d\sigma = 0, \end{aligned}\tag{26}$$

which holds for each $\mathbf{q} \in (H^1(\Omega))^4$, we next deduce that $\nabla\varphi^* = \mathbf{0}$ on $\partial\Omega$, that is $\varphi^* \in H_0^2(\Omega)$. Analogous considerations lead us to conclude that each couple $(\varphi, \boldsymbol{\psi}) \in \mathcal{V}$ is such that $\varphi \in H_0^2(\Omega)$ and $\boldsymbol{\psi} = -\mathbf{H}\varphi$.

Noticing that $(\varphi^*, \boldsymbol{\psi}^*)$ is also solution of the variational equations

$$\int_\Omega \boldsymbol{\psi}^* \cdot \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}]\, da = \int_\Omega f\varphi\, da, \quad \forall(\varphi, \boldsymbol{\psi}) \in \mathcal{V}, \tag{27}$$

which represent the optimality conditions of the functional $\mathcal{F}$, we finally find

$$\int_\Omega \mathbf{H}\varphi^* \cdot \boldsymbol{\mathcal{A}}[\mathbf{H}\varphi]\, da = \int_\Omega f\varphi\, da, \quad \forall \varphi \in H_0^2(\Omega), \tag{28}$$

and thus φ^* coincides with the minimizer φ_0 of the functional $\mathcal{E}$ defined as in Eq. (7).

3. THE LUMPED STRESS METHOD

Consider a double partition of the domain Ω, that is a *primal mesh*

$$\Pi_h = \{\Omega_m, m \in \{1, 2, \ldots, M\}\}, \tag{29}$$

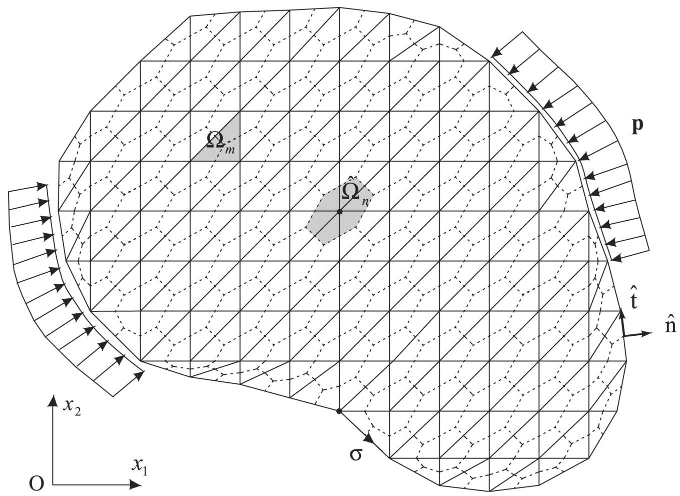

FIG. 1. Primal and dual meshes.

composed of triangular elements, and a *dual mesh*

$$\hat{\Pi}_h = \{\hat{\Omega}_n, n \in \{1, 2, \ldots, N\}\}, \tag{30}$$

formed by polygons which are built around each node of the primal mesh Π_h. We assume that dual polygons $\hat{\Omega}_n$ divide into two equal parts the edges of the primal triangles Ω_m (Figure 1).

Here and in what follows, the index h refers to the *mesh size*, defined as $h = \sup_{m \in \{1,2,\ldots,M\}}\{diam(\Omega_m)\}$, where $diam(\Omega_m) = \max\{|\mathbf{x} - \mathbf{y}|\,,\ \mathbf{x}, \mathbf{y} \in \Omega_m\}$.

We let S_h and T_h denote the space of piecewise linear scalar function defined over Π_h (*polyhedral functions*), and the space of piecewise constant tensor-valued functions defined over $\hat{\Pi}_h$, respectively. Moreover, we let S_{0h} denote the subspace of S_h consisting of polyhedral functions vanishing at the boundary of Ω, that is: $S_{0h} = S_h \cap H_0^1(\Omega)$.

Our numerical approach to the principle of minimum complementary energy (*Lumped Stress Method or LSM*), can be divided into two steps [13, 14].

First, for a given $\hat{\varphi} \in S_{0h}$, we solve the *pre-minimization* problem

Find $\hat{\boldsymbol{\psi}} = \hat{\boldsymbol{\psi}}(\hat{\varphi}) \in \mathcal{V}_{\hat{\varphi}}$ *such that*

$$U(\hat{\boldsymbol{\psi}}) = \inf_{\boldsymbol{\psi} \in \mathcal{V}_{\hat{\varphi}}} U(\boldsymbol{\psi}), \tag{31}$$

where

$$U(\boldsymbol{\psi}) = \frac{1}{2}\int_{\Omega} \boldsymbol{\psi} \cdot \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}]\, da, \tag{32}$$

and

$$\mathcal{V}_{\hat{\varphi}} = \{\boldsymbol{\psi} \in (L^2(\Omega))^4 / \beta((\hat{\varphi}, \boldsymbol{\psi}), \hat{\mathbf{q}}) = 0, \forall \hat{\mathbf{q}} \in T_h\}. \tag{33}$$

Next, we approach the unconstrained minimization problem

Find $\hat{\varphi}_h \in S_{0h}$ *such that*

$$\mathcal{E}_h(\hat{\varphi}_h) = \min_{\hat{\varphi}_h \in S_{0h}} \mathcal{E}_h(\hat{\varphi}), \tag{34}$$

where

$$\mathcal{E}_h(\hat{\varphi}) = U(\hat{\boldsymbol{\psi}}(\hat{\varphi})) - l(\hat{\varphi}). \tag{35}$$

Let us consider problem (31). Upon expressing a generic $\hat{\mathbf{q}} \in T_h$ as $\hat{\mathbf{q}} = \sum_{n=1}^{N} \hat{\mathbf{q}}(n)\, \chi_n$, χ_n being the characteristic function of $\hat{\Omega}_n$, we find (see the Appendix)

$$\begin{aligned}\beta((\hat{\varphi}, \boldsymbol{\psi}), \hat{\mathbf{q}}) = &- \sum_{n=1}^{N} \hat{\mathbf{q}}(n) \cdot \left(\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da + \int_{\hat{\Omega}_n} \boldsymbol{\psi}\, da \right) \\ &+ \sum_{b \in B} \hat{\mathbf{q}}(b) \cdot \int_{\gamma_b} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma, \\ &\forall\, ((\hat{\varphi}, \boldsymbol{\psi}), \hat{\mathbf{q}}) \in (S_h \times (L^2(\Omega))^4) \times T_h,\end{aligned} \tag{36}$$

where $\gamma_b = \partial\hat{\Omega}_b \cap \partial\Omega$, B denoting the set of the indices taken by the boundary nodes of the mesh Π_h. Notice that the quantity $\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da$ is well defined, since $\mathbf{H}\hat{\varphi}$ represents a linear Dirac delta distributed over the skeleton of Π_h.

Using Eq. (36) in the definition (31), we deduce that the elements of the space $\mathcal{V}_{\hat{\varphi}}$ are the functions $\boldsymbol{\psi} \in (L^2(\Omega))^4$ such that

$$\int_{\hat{\Omega}_n} \boldsymbol{\psi}\, da = \begin{cases} -\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da & \text{if } n \in I, \\ -\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da + \int_{\gamma_n} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma & \text{if } n \in B, \end{cases} \tag{37}$$

I denoting the set of the indices taken by the interior nodes of Π_h.

Now, consider that *Jensen's inequality* and the spectral decomposition of $\mathcal{A}$ yield

$$\begin{aligned}2U(\boldsymbol{\psi}) &= \sum_{n=1}^{N} \int_{\hat{\Omega}_n} \boldsymbol{\psi} \cdot \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}]\, da \\ &\geq \sum_{n=1}^{N} \frac{1}{ar(\hat{\Omega}_n)} \left(\int_{\hat{\Omega}_n} \boldsymbol{\psi}\, da \right) \left(\int_{\hat{\Omega}_n} \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}]\, da \right), \\ &\qquad \forall \boldsymbol{\psi} \in \{(L^2(\Omega))^4,\end{aligned} \tag{38}$$

where $ar(\hat{\Omega}_n)$ denotes the area of $\hat{\Omega}_n$. In particular, Eq. (38) holds with the sign of equality if $\boldsymbol{\psi} \in T_h$. Combining Eqs. (37) and (38), we deduce that the minimizer of U over $\mathcal{V}_{\hat{\varphi}}$ is the element $\hat{\boldsymbol{\psi}}$ of T_h such that

$$\hat{\boldsymbol{\psi}} = -\mathbf{H}_h\hat{\varphi} = -\sum_{n=1}^{N} \mathbf{H}_h\hat{\varphi}(n)\chi_n, \tag{39}$$

where

$$\mathbf{H}_h\hat{\varphi}(n) = \begin{cases} \dfrac{1}{ar(\hat{\Omega}_n)} \displaystyle\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da & \text{if } n \in I, \\ \dfrac{1}{ar(\hat{\Omega}_n)} \left(\displaystyle\int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da - \int_{\gamma_n} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma \right) & \text{if } n \in B. \end{cases} \tag{40}$$

Having solved problem (31), we are left with problem (34), where now we have

$$\mathcal{E}_h(\hat{\varphi}) = \frac{1}{2} \int_\Omega \mathbf{H}_h\hat{\varphi} \cdot \mathcal{A}[\mathbf{H}_h\hat{\varphi}]\, da - l(\hat{\varphi}). \tag{41}$$

By the positions (39)–(40) and the inequality of Eq. (38), it follows that $\mathcal{E}_h(\varphi) \leq \mathcal{E}(\varphi)$ for each $\varphi \in H_0^2(\Omega)$, $\mathcal{E}$ being defined as in Eq. (3). In particular, the functional $\mathcal{E}_h$ allows us to extend problem (6) to a functional space larger than $H^2(\Omega)$ including polyhedral stress functions. In this sense, we refer to the minimization of $\mathcal{E}_h$ as a *relaxation* of the original problem.

In order to prove the convergence of the LSM, it is useful to view the discrete problem of (34) as a suitable approximation of the mixed problem (19). As a matter of fact, our previous developments underlay that minimizing $\mathcal{E}_h$ over S_{0h} is equivalent to

Find $(\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h) \in \mathcal{W}_h$ such that

$$\mathcal{F}((\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h)) = \min_{(\hat{\varphi}, \hat{\boldsymbol{\psi}})\mathcal{W}_h} \mathcal{F}((\hat{\varphi}, \hat{\boldsymbol{\psi}})), \tag{42}$$

where $\mathcal{W}_h$ is the function space defined as

$$\mathcal{W}_h = \{(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in S_{0h} \times T_h \,/\, \beta((\hat{\varphi}, \hat{\boldsymbol{\psi}}), \hat{\mathbf{q}}) = 0\,, \quad \forall \hat{\mathbf{q}} \in T_h\}. \tag{43}$$

Actually problem (42) derives from an external approximation $\mathcal{W}_h$ of the space $\mathcal{V}$ defined as in Eq. (15), since the multiplier $\hat{\mathbf{q}}$ is chosen in T_h, which is not contained in $(H^1(\Omega))^4$. Nevertheless, T_h and the proper subspace $(S_h)^4$ of $(H^1(\Omega))^4$ can be put in 1-1 correspondence through the following linear mapping ϑ_h

$$\begin{cases} \vartheta_h\hat{\mathbf{q}} \in (S_h)^4, & \forall \hat{\mathbf{q}} = \displaystyle\sum_{n=1}^N \hat{\mathbf{q}}(n)\chi_n \in T_h, \\ \vartheta_h\hat{\mathbf{q}}(\mathbf{X}_n) = \hat{\mathbf{q}}(n), & \forall n \in \{1, 2, \ldots, N\}, \end{cases} \tag{44}$$

$\mathbf{x}_n$ being the position vector of the nth node of the primal mesh. In particular, given an arbitrary $(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in S_{0h} \times T_h$ and an arbitrary $\hat{\mathbf{q}} \in T_h$, it is easy to verify that (see the Appendix)

$$\beta((\hat{\varphi}, \hat{\boldsymbol{\psi}}), \vartheta_h\hat{\mathbf{q}}) = \beta((\hat{\varphi}, \hat{\boldsymbol{\psi}}), \hat{\mathbf{q}}) + \mathrm{O}(\hat{\boldsymbol{\psi}}, \vartheta_h\hat{\mathbf{q}}), \tag{45}$$

where

$$\left|\mathrm{O}(\hat{\boldsymbol{\psi}}, \vartheta_h\hat{\mathbf{q}})\right| \leq h\, \|\hat{\boldsymbol{\psi}}\|_0\, |\vartheta_h\hat{\mathbf{q}}|_1\,. \tag{46}$$

Thus, the approximation space (43) can be also defined as

$$\mathcal{W}_h = \big\{(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in S_{0h} \times T_h \,/\, \beta((\hat{\varphi}, \hat{\boldsymbol{\psi}}), \hat{\mathbf{q}}') = \mathrm{O}(\hat{\boldsymbol{\psi}}, \hat{\mathbf{q}}'), \quad \forall \hat{\mathbf{q}}' \in (S_h)^4\big\}, \tag{47}$$

and problem (42) can be regarded as an internal approximation of the continuous problem (18), associated with a relaxation of the constraint equation $\beta((\cdot, \cdot), \cdot) = 0$.

4. EXISTENCE AND UNIQUENESS

Arguing as in Theorem 1, it is not difficult to prove the existence and the uniqueness of problem (42). We address this question to Theorem 2.

Theorem 2. *The discrete problem (42) has one and only one solution* $(\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h)$.

Proof. Define the norm

$$\|(\hat{\varphi}, \hat{\boldsymbol{\psi}})\|_{\mathcal{W}_h} = \left(|\hat{\varphi}|_1^2 + \|\hat{\boldsymbol{\psi}}\|_0^2\right)^{1/2}, \tag{48}$$

and observe that the couple $(\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h)$ is also solution of the variational equations

$$a((\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h), (\hat{\varphi}, \hat{\boldsymbol{\psi}})) = \ell((\hat{\varphi}, \hat{\boldsymbol{\psi}}))\,, \quad \forall (\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h. \tag{49}$$

That is,

$$\int_\Omega \hat{\boldsymbol{\psi}} \cdot \mathcal{A}[\boldsymbol{\psi}_h]\, da = \int_\Omega f\, \hat{\varphi}\, da\,, \quad \forall (\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h. \tag{50}$$

Given an arbitrary $(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h$, by choosing $\hat{\mathbf{q}}' = \hat{\varphi}\,(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2)$ in Eq. (47) and taking Eq. (46) into account, we obtain

$$\left|\int_\Omega \nabla\hat{\varphi} \cdot \nabla\hat{\varphi}\, da - \int_\Omega (\hat{\psi}_{11} + \hat{\psi}_{22})\hat{\varphi}\, da\right| \leq \sqrt{2}\, h\, \|\hat{\boldsymbol{\psi}}\|_0\, |\hat{\varphi}|_1\,, \quad \forall (\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h, \tag{51}$$

from which it is easy to deduce

$$|\hat{\varphi}|_1 \leq \sqrt{2}(C(\Omega) + h)\|\hat{\boldsymbol{\psi}}\|_0, \quad \forall (\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h, \tag{52}$$

and

$$\|(\hat{\varphi}, \hat{\boldsymbol{\psi}})\|_{\mathcal{W}_h}^2 \leq \alpha(h) \int_\Omega \hat{\boldsymbol{\psi}} \cdot \mathcal{A}[\hat{\boldsymbol{\psi}}]\, da, \quad \forall (\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h, \tag{53}$$

where

$$\alpha(h) = \frac{1 + 2(C(\Omega) + h)^2}{\Lambda_{\min}}. \tag{54}$$

Thus, the bilinear form a is coercive on $\mathcal{W}_h$. Since, on the other hand, it is easy to prove that both a and the linear form ℓ

given by Eq. (24) are continuous on $\mathcal{W}_h$ (see Theorem 1), the thesis follows from the Lax-Milgram lemma. ■

5. ERROR ESTIMATES

We can regard $(\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h)$ as a family of approximate solutions of the minimization problem (19), since each value of h is associated with a problem (42). Our present objective is to prove that there exist families of solutions $(\hat{\varphi}_h, \hat{\boldsymbol{\psi}}_h)$ which *converge to the exact solution* $(\varphi^*, \boldsymbol{\psi}^*)$ of problem (19), in the sense that $e_h = |\varphi_0 - \hat{\varphi}_h|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_h\|_0 \to 0$ as $h \to 0$. Moreover, upon assuming appropriate smoothness properties on $(\varphi^*, \boldsymbol{\psi}^*)$, we wish to find the rate of convergence of such families of solutions, that is a real number r with the property that there exists a constant $C((\varphi^*, \boldsymbol{\psi}^*))$ independent of h such that $e_h \le C((\varphi^*, \boldsymbol{\psi}^*))h^r$. We recall that, by Theorem 1, φ^* coincides with the solution φ_0 of problem (6), and $\boldsymbol{\psi}^*$ coincides with $-\mathbf{H}\varphi_0$. For further use, we set

$$\boldsymbol{\psi}_0 = -\mathbf{H}\varphi_0,\ \mathbf{q}_0 = \boldsymbol{\mathcal{A}}[\boldsymbol{\psi}_0] = -\boldsymbol{\mathcal{A}}[\mathbf{H}\varphi_0]. \tag{56}$$

We begin by proving the following Lemma, that gives an abstract estimate of the *error* e_h.

Lemma 1. *Suppose that the solution φ_0 of problem (6) belongs to the space $H^m(\Omega) \cap H_0^2(\Omega)$, with $m \ge 3$. Then, for h sufficiently small, there exist positive constants c_1, c_2, c_3 independent of h such that*

$$\begin{aligned} e_h &= |\varphi_0 - \hat{\varphi}_h|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_h\|_0 \\ &\le c_1 \inf_{(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h} \left(|\varphi_0 - \hat{\varphi}|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}\|_0\right) \\ &\quad + c_2 \inf_{\hat{\mathbf{q}}' \in (S_h)^4} \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 + c_3 h \, \|\mathbf{q}_0\|_1 . \end{aligned} \tag{57}$$

Proof. From Theorem 1, we know that $(\varphi_0, \boldsymbol{\psi}_0)$ is solution of the variational equation

$$\begin{aligned} &a((\varphi_0, \boldsymbol{\psi}_0),\ (\varphi, \boldsymbol{\psi})) + \beta((\varphi, \boldsymbol{\psi}),\ \mathbf{q}_0) \\ &\quad = \ell((\varphi, \boldsymbol{\psi})), \quad \forall (\varphi, \boldsymbol{\psi}) \in H_0^1(\Omega) \times (L^2(\Omega))^4, \end{aligned} \tag{58}$$

that is, using the definitions (16), (17) and (21)

$$\int_\Omega div\, \mathbf{q}_0 \cdot \nabla \varphi \, da = \int_\Omega f \varphi \, da, \qquad \forall \varphi \in H_0^1(\Omega). \tag{59}$$

Now, let $(\hat{\varphi}, \hat{\boldsymbol{\psi}})$ be an arbitrary element of the space $\mathcal{W}_h$ defined as in Eq. (47), and let $\hat{\mathbf{q}}'$ be an arbitrary element of the space $(S_h)^4$. From Eq. (59) and by Eq. (47) one gets

$$\begin{aligned} &\beta((\hat{\varphi}_h - \hat{\varphi}, \hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}), \mathbf{q}_0 - \hat{\mathbf{q}}') \\ &\quad = \beta((\hat{\varphi}_h - \hat{\varphi}, \hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}), \mathbf{q}_0) - \beta((\hat{\varphi}_h - \hat{\varphi}, \hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}), \hat{\mathbf{q}}') \\ &\quad = \int_\Omega \nabla(\hat{\varphi}_h - \hat{\varphi}) \cdot div\mathbf{q}_0 \, da - \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot \mathbf{q}_0 \, da \\ &\qquad + \mathrm{O}(\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}, \hat{\mathbf{q}}') \\ &\quad = \int_\Omega f(\hat{\varphi}_h - \hat{\varphi}) \, da - \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot \mathbf{q}_0 \, da + \mathrm{O}(\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}, \hat{\mathbf{q}}') \\ &\quad = \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}_h] - \mathbf{q}_0) \, da + \mathrm{O}(\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}, \hat{\mathbf{q}}'). \end{aligned} \tag{60}$$

On the other hand, by Eq. (60), the continuity of the bilinear form β, which is easy to prove, the definition (20) and the estimates of Eqs. (46), (52), it follows

$$\begin{aligned} &\left| \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}_h] - \mathbf{q}_0) \, da \right| \\ &\quad \le \gamma(h) \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 + h \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 |\hat{\mathbf{q}}'|_1, \end{aligned} \tag{61}$$

where

$$\gamma(h) = M(1 + \sqrt{2}(C(\Omega) + h)), \tag{62}$$

M being a positive constant independent of h.

The estimate of Eq. (61) and the Cauchy-Schwartz inequality allow us to write

$$\begin{aligned} \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0^2 &\le \frac{1}{\Lambda_{\min}} \left(\left| \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}_h] - \mathbf{q}_0) \, da \right| \right. \\ &\quad \left. + \left| \int_\Omega (\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}) \cdot (\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}] - \mathbf{q}_0) \, da \right| \right) \\ &\le \frac{\gamma(h)}{\Lambda_{\min}} \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 + \frac{h}{\Lambda_{\min}} \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 |\hat{\mathbf{q}}'|_1 \\ &\quad + \frac{1}{\Lambda_{\min}} \|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 \|\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}] - \mathbf{q}_0\|_0. \end{aligned} \tag{63}$$

Therefore, since

$$|\hat{\mathbf{q}}'|_1 \le \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 + \|\mathbf{q}_0\|_1, \tag{64}$$

$$\|\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}}] - \mathbf{q}_0\|_0 = \|\boldsymbol{\mathcal{A}}[\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}_0]\|_0 \le \Lambda_{\max} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}_0\|_0, \tag{65}$$

for $h \le 1$ it results

$$\|\hat{\boldsymbol{\psi}}_h - \hat{\boldsymbol{\psi}}\|_0 \le k_1 \|\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\|_0 + k_2 \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 + k_3 h \|\mathbf{q}_0\|_1, \tag{66}$$

k_1, k_2, k_3 being positive constants independent of h.

Now, consider that, by the triangle inequality and the inequality of Eq. (52), one gets

$$\begin{aligned} &|\varphi_0 - \hat{\varphi}_h|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_h\|_0 \\ &\quad \le |\varphi_0 - \hat{\varphi}|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}\|_0 \\ &\qquad + \left(1 + \sqrt{2}(C(\Omega + h)\right) \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}_h\|_0 \end{aligned} \tag{67}$$

$(\hat{\varphi}, \hat{\boldsymbol{\psi}})$ being an arbitrary element of $\mathcal{W}_h$. Upon combining inequalities (66)–(67), the estimate of Eq. (57) follows. ■

In order to apply the abstract estimate Eq. (57), we need to consider families of primal and dual meshes having some regularity and uniformity properties.

In particular, we need to consider a family of triangulations Π_h which is *regular affine* in the following sense (see, e.g., Ciarlet [3])

(H1): *i) There exist a constant* σ *independent of h such that* $h/\rho_h \le \sigma$, *where* $\rho_h = \inf_{m\in\{1,2,\dots,M\}}$ sup *(diameters of all circles contained in* $\Omega_m \in \Pi_h$*); ii) the mesh size h approaches zero.*

(H2): *all the triangles* $\Omega_m \in \Pi_h$, *are affine-equivalent to a single reference triangle, for all h.*

Introduce now the basis $\{g_1, g_2, \dots, g_N\}$ of S_h such that $g_n(\mathbf{x}_j) = \delta_{nj}$ (δ_{nj} being the *Kronecker symbol*), and denote the support of the generic function g_n (i.e., the union of the triangles having a vertex at $\mathbf{x}_n$) by G_n.

We say that a node $\mathbf{x}_n$ of Π_h owes the property $(\mathcal{P}_\Sigma)$ if, given an arbitrary tensor $\mathbf{H}$ (independent of position), it results

$$\sum_{j\in\mathcal{I}_n}\int_{G_n}\mathbf{H}(\mathbf{x}-\mathbf{x}_j)\cdot(\mathbf{x}-\mathbf{x}_j)\nabla g_j\otimes\nabla g_n = \mathbf{0}, \tag{68}$$

$\mathcal{I}_n$ being the set of the indices taken by the nodes of Π_h lying on the closure $\bar{G}_n$ of G_n (that is n and the nodes connected to n by an edge of Π_h).

It is not difficult to verify that such a property, generalizing a similar one formulated by Glowinski in [1], holds in the following remarkable cases:

(1) G_n is an hexagon or half an hexagon generated by a rectangular grid of nodes, which is uniform at least in one direction (see Figure 2);
(2) G_n is an hexagon or half an hexagon formed by triplets of equal isosceles triangles (see Figure 3).

The last two assumptions we need to introduce about primal and dual meshes are the following

(H3): *The boundaries of the polygons* $\hat{\Omega}_n \in \hat{\Pi}_h$ *are obtained by connecting the middle points of the sides of the triangles* $\Omega_m \in \Pi_h$ *having a vertex at* $\mathbf{x}_n$ *with the centroids of the same triangles.*

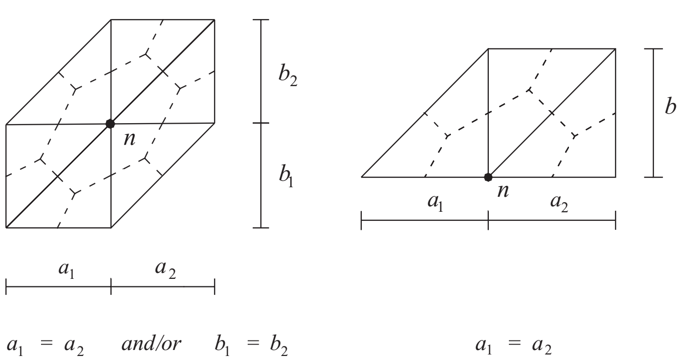

FIG. 2. Element G_n coincident with an hexagon or half an hexagon generated by a rectangular grid of nodes.

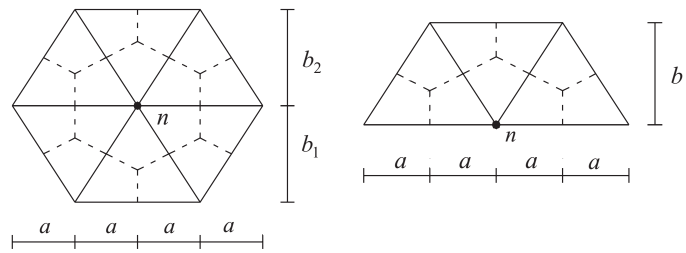

FIG. 3. Element G_n coincident with a hexagon or half a hexagon formed by triplets of equal isosceles triangles.

(H4): *For all h,* $\hat{\Pi}_h$ *can be divided in two disjointed parts* $\hat{\Pi}_{h_1} = \{\hat{\Omega}_j / j \in \mathcal{J}_1\}$ *and* $\hat{\Pi}_{h_2} = \{\hat{\Omega}_j / j \in \mathcal{J}_2\}$, *such that*

i) the elements of $\hat{\Pi}_{h_1}$ *are built around nodes owing the* $(\mathcal{P}_\Sigma)$ *property;*

ii) $ar(\Omega_{h_2}) = \sum_{j\in\mathcal{J}_2} ar(\hat{\Omega}_j) \to 0$ *as* $h \to 0$.

We set $ar(\Omega_{h_1}) = \sum_{j\in\mathcal{J}_1} ar(\hat{\Omega}_j)$. It is worthwhile noticing that (H4) holds, for example, when the *core* of $\hat{\Pi}_h$ is formed by elements centered at nodes owing the $(\mathcal{P}_\Sigma)$ property, and $\hat{\Pi}_{h_2}$ coincides with a strip of elements adjacent to the boundary of Ω. In this case, $ar(\Omega_{h_2})$ is of $O(h)$ (see, e.g., Figure 1). Another remarkable case, interesting for the applications presented in Part II, is that of a rectangular domain covered by a uniform rectangular grid of nodes. Here, all the nodes have the $(\mathcal{P}_\Sigma)$ property, with exception to the four corner nodes, and thus $ar(\Omega_{h_2})$ is of $O(h^2)$.

The assumption (H3) allows us to express the constraint of (39) in a different form. Indeed, consider that is possible to write $\vartheta_h\hat{\mathbf{q}}(\mathbf{x}) = \sum_{n=1}^{N}\hat{\mathbf{q}}(n)g_n(\mathbf{x}), \quad \forall\hat{\mathbf{q}} \in T_h$. Hence, given an arbitrary $(\hat{\varphi}, \hat{\psi}) \in S_{0h} \times T_h$, enforcing the variational equation (see the Appendix)

$$\beta((\hat{\varphi},\hat{\psi}),\hat{\mathbf{q}}) = \int_\Omega \nabla\hat{\varphi}\cdot div\,\vartheta_h\hat{\mathbf{q}}\,da - \int_\Omega \hat{\psi}\cdot\hat{\mathbf{q}}\,da = 0, \quad \forall\hat{\mathbf{q}}\in T_h, \tag{69}$$

we easily find

$$\hat{\psi}(n) = \frac{1}{ar(\hat{\Omega})_n}\int_\Omega \nabla\hat{\varphi}\otimes\nabla g_n\,da = \frac{\int_\Omega \nabla\hat{\varphi}\otimes\nabla g_n\,da}{\int_\Omega g_n\,da}, \quad \forall n\in\{1,2,\dots,N\}, \tag{70}$$

since $\int_\Omega g_n da = ar(G_n)/3$,and $ar(\hat{\Omega}_n) = ar(G_n)/3$ in virtue of (H3). Clearly, the integrals on the right-hand side of (70) can be restricted to G_n.

The following Lemma 2 and Theorem 3 give us the desired estimate of the error e_h. The result we find is similar to those given by Scholtz in [4] and by Davini and Pitacco in [7] for the biharmonic problem. Use is made of the notation $\|\cdot\|_{m,\infty}$ and $|\cdot|_{m,\infty}$ for the norm and the seminorm in the Sobolev space $W^{m,\infty}(\Omega)$, respectively.

Lemma 2. *Assume that (H1), (H2), (H3) and (H4) hold, and that the solution* φ_0 *of problem (6) belongs to the space*

$W^{3,\infty}(\Omega) \cap H_0^2(\Omega)$. Then, there exists a couple $(\hat{\varphi}_0, \hat{\psi}_0) \in \mathcal{W}_h$ and constants $\bar{c}_1, \bar{c}_2$ and $\bar{c}_3$ independent of h such that

$$|\varphi_0 - \hat{\varphi}_0|_1 \leq \bar{c}_1 h |\varphi_0|_2, \tag{71}$$

$$\|\psi_0 - \hat{\psi}_0\|_0 \leq \bar{c}_2 h \sqrt{ar(\Omega_{h_1})} |\varphi_0|_{3,\infty} + \bar{c}_3 h \sqrt{ar(\Omega_{h_2})} |\varphi_0|_{2,\infty}. \tag{72}$$

Proof. Consider the linear mappings $r_h : H^1(\Omega) \to S_h$ and $R_h : (L^2(\Omega))^4 \to T_h$ defined as

$$r_h \varphi = \sum_{n=1}^{N} \varphi(\mathbf{x}_n) g_n, \quad \forall \varphi \in H^1(\Omega), \tag{73}$$

$$R_h \psi = \sum_{n=1}^{N} \frac{\int_\Omega \psi g_n \, da}{\int_\Omega g_n \, da} \chi_n, \quad \forall \psi \in (L^2(\Omega))^4. \tag{74}$$

Since r_h leaves invariant piecewise linear functions on Π_h, by (H1), (H2), the properties of projection operators, and the Poincaré inequality, we get

$$|\varphi_0 - r_h \varphi_0|_1 \leq k\, h |\varphi_0|_2, \tag{75}$$

where k is a constant independent of h.

On the other hand, it is easy to show (cf. [7]) that there exists a constant k' independent of h such that

$$\|\psi_0 - R_h \psi_0\|_0 \leq k'\, h |\psi_0|_1 = k'\, h |\varphi_0|_3. \tag{76}$$

Now, consider the couple $(\hat{\varphi}_0, \hat{\psi}_0) \in \mathcal{W}_h$ with $\hat{\varphi}_0 = r_h \varphi_0$ and $\hat{\psi}_0 = \sum_{n=1}^{N} \hat{\psi}_0(n)\, \chi_n$ such that

$$\psi_0(n) = \frac{1}{ar(\hat{\Omega}_n)} \int_\Omega \nabla \hat{\varphi}_0 \otimes \nabla g_n \, da, \quad \forall n \in \{1, 2, \ldots, N\}. \tag{77}$$

Due to the assumption $\varphi_0 \in W^{3,\infty}(\Omega)$ and the embedding $W^{3,\infty}(\Omega) \to C^2(\Omega)$ (see Adams [19]), we can apply the following Taylor's formula

$$\begin{aligned} \varphi_0(\mathbf{x}_n) = \varphi_0(\mathbf{x}) &+ \nabla \varphi_0(\mathbf{x}) \cdot (\mathbf{x}_n - \mathbf{x}) \\ &+ \frac{1}{2} \mathbf{H} \varphi_0(\bar{\mathbf{x}}_n)(\mathbf{x}_n - \mathbf{x}) \cdot (\mathbf{x}_n - \mathbf{x}), \\ &\qquad \forall n \in \{1, 2, \ldots, N\}, \end{aligned} \tag{78}$$

where $\mathbf{x}$ is an arbitrary point of G_n and $\bar{\mathbf{x}}_n = \bar{\mathbf{x}}_n(\mathbf{x})$ is an interior point of the segment $\mathbf{x}_n - \mathbf{x}$. Making use of Eq. (78) and observing that $\nabla \hat{\varphi}_0(\mathbf{x}) = \sum_{j \in \mathcal{I}_n} \varphi_0(\mathbf{x}_j)\, \nabla g_j(\mathbf{x}),\ \forall \mathbf{x} \in G_n$, we obtain

$$\begin{aligned} \nabla \hat{\varphi}_0(\mathbf{x}) = \sum_{j \in \mathcal{I}_n} \Big(&\varphi_0(\mathbf{x}) + \nabla \varphi_0(\mathbf{x}) \cdot (\mathbf{x}_j - \mathbf{x}) \\ &+ \frac{1}{2} \mathbf{H} \varphi_0(\bar{\mathbf{x}}_j)(\mathbf{x}_j - \mathbf{x}) \cdot (\mathbf{x}_j - \mathbf{x}) \Big) \nabla g_j(\mathbf{x}). \end{aligned} \tag{79}$$

On the other hand, the base functions g_j have the property that $\sum_{j \in \mathcal{I}_n} g_j(\mathbf{x}) = 1,\ \forall \mathbf{x} \in G_n$. Thus, it results $\sum_{j \in \mathcal{I}_n} \nabla g_j(\mathbf{x}) = \mathbf{0}$, $\forall \mathbf{x} \in G_n$, and

$$\begin{aligned} &\sum_{j \in \mathcal{I}_n} (\nabla \varphi_0(\mathbf{x}) \cdot (\mathbf{x}_j - \mathbf{x})) \nabla g_j(\mathbf{x}) \\ &\quad = \sum_{j \in \mathcal{I}_n} (\nabla \varphi_0(\mathbf{x}) \cdot \mathbf{x}_j) \nabla g_j(\mathbf{x}) - (\nabla \varphi_0(\mathbf{x}) \cdot \mathbf{x}) \sum_{j \in \mathcal{I}_n} \nabla g_j(\mathbf{x}) \\ &\quad = \nabla^T \Bigg(\sum_{j \in \mathcal{I}_n} g_j(\mathbf{x}) \mathbf{x}_j \Bigg) \nabla \varphi_0(\mathbf{x}) = \nabla \varphi_0(\mathbf{x}), \end{aligned} \tag{80}$$

since $\sum_{j \in \mathcal{I}_n} g_j(\mathbf{x})\, \mathbf{x}_j = \mathbf{x}$, and $\nabla^T \mathbf{x} = \mathbf{I}$, $\mathbf{I}$ being the identity tensor. Therefore, Eq. (79) can be rewritten as

$$\begin{aligned} \nabla \hat{\varphi}_0(\mathbf{x}) = \nabla \varphi_0(\mathbf{x}) &+ \frac{1}{2} \sum_{j \in \mathcal{I}_n} (\mathbf{H} \varphi_0(\bar{\mathbf{x}}_j)(\mathbf{x}_j - \mathbf{x}) \\ &\cdot (\mathbf{x}_j - \mathbf{x})) \nabla g_j(\mathbf{x}), \qquad \forall \mathbf{x} \in G_n. \end{aligned} \tag{81}$$

Upon substituting Eq. (81) into Eq. (77), we obtain

$$\begin{aligned} \hat{\psi}_0(n) = &\frac{1}{ar(\hat{\Omega}_n)} \int_\Omega \nabla \hat{\varphi}_0 \otimes \nabla g_n \, da \\ &+ \frac{1}{2ar(\hat{\Omega}_n)} \sum_{j \in \mathcal{I}_n} \int_\Omega (\mathbf{H} \varphi_0(\bar{\mathbf{x}}_j)(\mathbf{x}_j - \mathbf{x}) \\ &\cdot (\mathbf{x}_j - \mathbf{x})) \nabla g_j \otimes \nabla g_n \, da, \quad \forall n \in \{1, 2, \ldots, N\}. \end{aligned} \tag{82}$$

Notice that the Green formula gives

$$\int_\Omega \nabla \varphi_0 \otimes \nabla g_n \, da = \int_{G_n} \nabla \varphi_0 \otimes \nabla g_n \, da = \int_{G_n} \psi_0 g_n \, da, \tag{83}$$

since either $\nabla \varphi_0$ or g_n are zero on ∂G_n. Taking into account Eq. (83), the definition (74) and considering that (H3) implies $\int_\Omega g_n da = ar(G_n)/3 = ar(\hat{\Omega}_n)$, we can rewrite Eq. (82) as follows

$$\begin{aligned} \hat{\psi}_0(n) = R_h \psi_0(n) &+ \frac{1}{2ar(\hat{\Omega}_n)} \sum_{j \in \mathcal{I}_n} \int_\Omega (\mathbf{H} \varphi_0(\bar{\mathbf{x}}_j)(\mathbf{x}_j - \mathbf{x}) \\ &\cdot (\mathbf{x}_j - \mathbf{x})) \nabla g_j \otimes \nabla g_n \, da, \quad \forall n \in \{1, 2, \ldots, N\}. \end{aligned} \tag{84}$$

Here, $|\mathbf{x}_j - \mathbf{x}| \leq h$, and $|\nabla g_j| \leq ch^{-1}, \forall j \in \{1, 2, \ldots, N\}$. Thus, from Hölder inequality, one deduces

$$|\hat{\boldsymbol{\psi}}_0(n) - R_h \boldsymbol{\psi}_0(n)| \leq k'' |\varphi_0|_{2,\infty}, \quad \forall n \in \{1, 2, \ldots, N\}, \quad (85)$$

k'' being a constant independent of h.

A refinement of the estimate (85) can be obtained by considering nodes owing the $(\mathcal{P}_\Sigma)$ property. Indeed, define the *difference quotient* of $\mathbf{H}\varphi_0$ in the direction of $\hat{\mathbf{e}}_\alpha$ as

$$D_\alpha^{\bar{h}} \mathbf{H}\varphi_0(\mathbf{x}) = \frac{\mathbf{H}\varphi_0(\mathbf{x} + \bar{h}\hat{\mathbf{e}}_\alpha) - \mathbf{H}\varphi_0(\mathbf{x})}{\bar{h}}, \quad (86)$$

and recall the standard estimate

$$\left\| D_\alpha^{\bar{h}} \mathbf{H}\varphi_0 \right\|_{L^\infty(G'_n)} \leq \|\nabla \mathbf{H}\varphi_0\|_{L^\infty(G'_n)} \leq |\varphi_0|_{3,\infty}, \quad (87)$$

which holds for any $G'_n \subset\subset G_n$ and any $\bar{h} < dist(G'_n, \partial\, G_n)$ (see, e.g., Renardy and Rogers [20]). Since we may express $\bar{\mathbf{x}}_j(\mathbf{x})$ as $\mathbf{x}_n + \bar{h}_{j1}(\mathbf{x})\, \hat{\mathbf{e}}_1 + \bar{h}_{j2}(\mathbf{x})\, \hat{\mathbf{e}}_2, \forall j \in \mathcal{I}_n$, where $\bar{h}_{j\alpha}(\mathbf{x}) < h$, it results

$$\mathbf{H}\varphi_0(\bar{\mathbf{x}}_j(\mathbf{x})) = \mathbf{H}\varphi_0(\mathbf{x}_n) + D_1^{\bar{h}_{j1}(\mathbf{x})} \mathbf{H}\varphi_0(\mathbf{x}_n) \bar{h}_{j1}(\mathbf{x}) + D_2^{\bar{h}_{j2}(\mathbf{x})} \mathbf{H}\varphi_0(\mathbf{x}_n) \bar{h}_{j2}(\mathbf{x}). \quad (88)$$

Thus, from Eqs. (84), (87), the definition (68) and the property (H4) we get

$$|\hat{\boldsymbol{\psi}}_0(n) - R_h \boldsymbol{\psi}_0(n)| \leq k'' |\varphi_0|_{3,\infty}, \quad \forall n \in \mathcal{J}_1. \quad (89)$$

The inequalities of Eqs. (85) and (89) yield

$$\begin{aligned} \|\hat{\boldsymbol{\psi}}_0 - R_h \boldsymbol{\psi}_0\|_0^2 &= \sum_{n \in \mathcal{J}_1 \cup \mathcal{J}_2} |\hat{\boldsymbol{\psi}}_0(n) - R_h \boldsymbol{\psi}_0(n)|^2 ar(\hat{\Omega}_n) \\ &\leq k'' \left(h^2 ar(\Omega_{h_1}) |\varphi_0|^2_{3,\infty} + ar(\Omega_{h_2}) |\varphi_0|^2_{2,\infty} \right). \quad (90) \end{aligned}$$

In conclusion, by applying Eq. (75), the triangle inequality

$$\|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_0\|_0 \leq \|\boldsymbol{\psi}_0 - R_h \boldsymbol{\psi}_0\|_0 + \|R_h \boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_0\|_0, \quad (91)$$

Eqs. (76) and (90), we get the proof of the thesis. ■

Theorem 3. *Let the properties of (H1), (H2), (H3) and (H4) hold, and let the solution* φ_0 *of problem (6) belong to* $H^4(\Omega) \cap W^{3,\infty}(\Omega) \cap H_0^2(\Omega)$. *Suppose further that it results in*

$$ar(\Omega_{h_2}) \leq ch, \quad (92)$$

where c is a constant independent of h.

Then, there exist constants C_1 *and* C_2 *independent of* φ_0 *and h such that*

$$e_h = |\varphi_0 - \hat{\varphi}_0|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}_0\|_0 \leq C_1 h \|\varphi_0\|_{3,\infty} + C_2 h^{\frac{1}{2}} \|\varphi_0\|_4. \quad (93)$$

Likely, when $\varphi_0 \in W^{4,\infty}(\Omega) \cap H_0^2(\Omega)$ *and in addition*

$$ar(\Omega_{h_2}) \leq ch^2, \quad (94)$$

it results

$$e_h \leq Ch \|\varphi_0\|_{4,\infty}, \quad (95)$$

with C independent of φ_0 *and h.*

Proof. Recall the abstract estimate (57) and observe that (H1) and (H2) imply that there exists a $\bar{c}$ independent of φ_0 and h such that [18, 21]

$$\inf_{\hat{\mathbf{q}}' \in (S_h)^4} \|\mathbf{q}_0 - \hat{\mathbf{q}}'\|_1 \leq \bar{c} h |\mathbf{q}_0|_2 \leq \bar{c} h \Lambda_{\max} \|\varphi_0\|_4. \quad (96)$$

On the other hand, from Lemma 2 it descends

$$\begin{aligned} \inf_{(\hat{\varphi}, \hat{\boldsymbol{\psi}}) \in \mathcal{W}_h} (|\varphi_0 - \hat{\varphi}|_1 + \|\boldsymbol{\psi}_0 - \hat{\boldsymbol{\psi}}\|_0) \leq \\ \bar{c}_1 h |\varphi_0|_2 + \bar{c}_2 h \sqrt{ar(\Omega_{h_1})} |\varphi_0|_{3,\infty} \\ + \bar{c}_3 \sqrt{ar(\Omega_{h_2})} |\varphi_0|_{2,\infty}. \quad (97) \end{aligned}$$

Finally, it is easy to recognize that

$$\|\mathbf{q}_0\|_1 \leq \Lambda_{\max} \|\varphi_0\|_3. \quad (98)$$

Upon substituting Eqs. (96)–(98) into Eq. (57) and taking into account the embeddings $H^4(\Omega) \to W^{2,\infty}(\Omega)$ and $W^{3,\infty}(\Omega) \to H^3(\Omega)$ [16], it follows that

$$e_h \leq c'_1 h \sqrt{ar(\Omega_{h_1})} \|\varphi_0\|_{3,\infty} + c'_2 \left(h + \sqrt{ar(\Omega_{h_2})} \right) \|\varphi_0\|_4, \quad (99)$$

where c'_1 and c'_2 are independent of φ_0 and h. The insertion of Eq. (92) into Eq. (99) gives the estimate of Eq. (93), for $h \leq 1$. Similarly, the insertion of Eq. (94) into Eq. (99) and the embedding $W^{4,\infty}(\Omega) \to H^4(\Omega)$ give the estimate of Eq. (95). ■

6. CONCLUDING REMARKS

The physical meaning of the Lumped Stress Method is the following. Consider an arbitrary $\hat{\varphi} \in S_{0h}$ defined as in Section 3, a latticed structure $\mathcal{B}_h$ coincident with the skeleton Σ_h of the primal mesh Π_h, and the stress field $\hat{\mathbf{T}} = \mathbf{W}^T \mathbf{H}\hat{\varphi}\mathbf{W}$. The latter consists of linear Dirac deltas with support Σ_h.

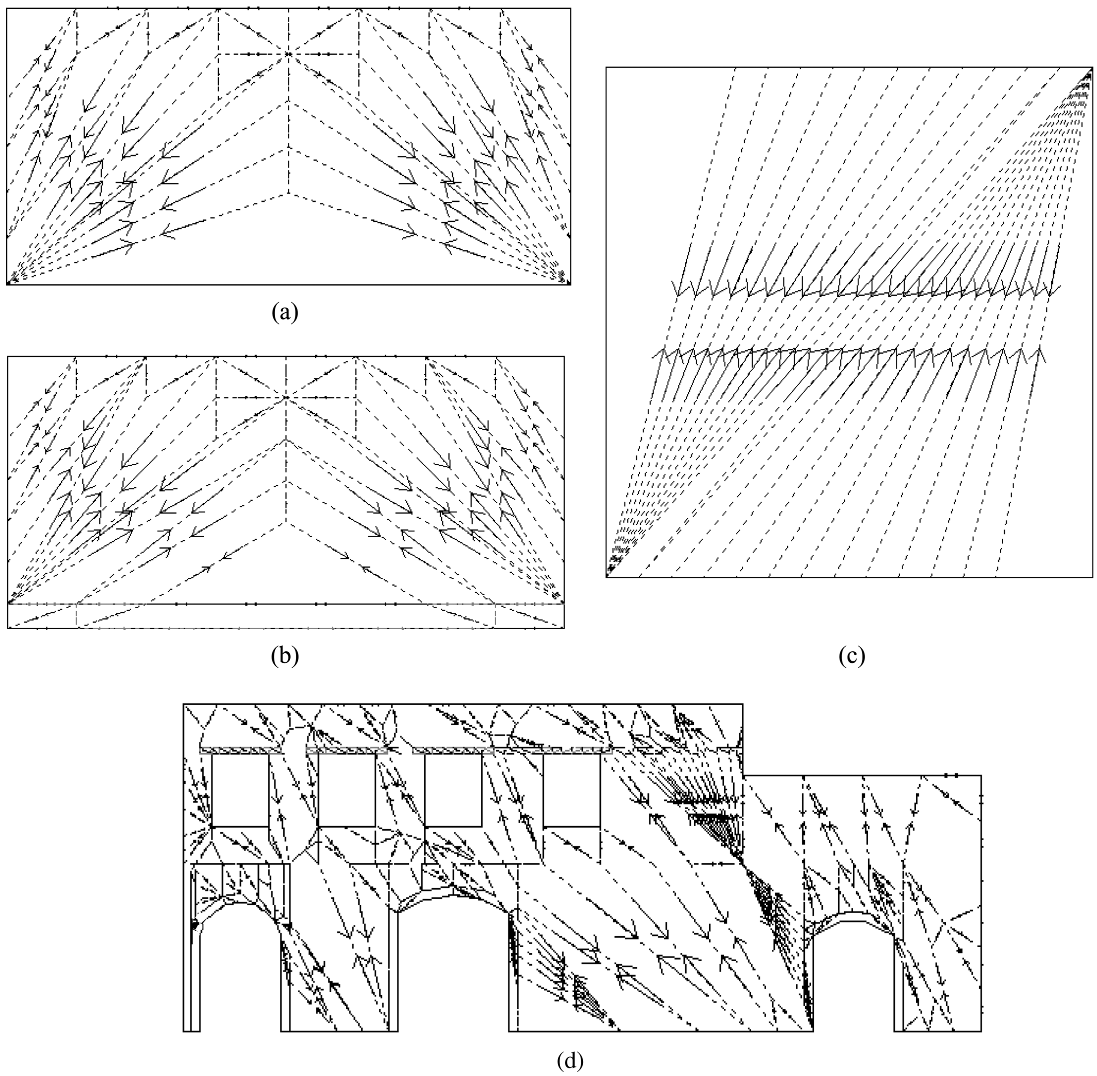

FIG. 4(a)–(d). LSM force networks obtained for several no-tension bodies.

It is easy to realize that the line integral of $\hat{\mathbf{T}}$ through each edge of Σ_h is a uniaxial tensor, which can be regarded as the axial force carried by the corresponding bar of $\mathcal{B}_h$.

The LSM approximates the stress in the neighborhood of each dual element by the quantity $\mathbf{T}^* + \hat{\mathbf{T}}_h(n)$, with $\hat{\mathbf{T}}_h(n) = \mathbf{W}^T \mathbf{H}_h \hat{\varphi}(n) \mathbf{W}$. Equation (40) shows that $\hat{\mathbf{T}}_h(n)$ coincides with a suitable composition of the uniaxial stresses carried by the bars of $\mathcal{B}_h$ incident to n.

It is useful to regard the quantity $\mathcal{E}_h(\hat{\varphi})$, defined as in Eq. (41), as the complementary energy of the truss $\mathcal{B}_h$.

Several applications of the LSM to technical problems and benchmark examples of 2D elasticity have been presented in [11, 12]. The particular ability of such a method in dealing with no-tension (*masonry-like*) materials has been illustrated in [16, 17].

Figures 4a–d show the LSM force networks for several elastic problems dealing with materials which do not react in tension. They refer to a transversally loaded clamped beam (Figure 4a); the same beam reinforced with a steel element at the bottom side (Figure 4b); a panel undergoing simple shear (Figure 4c); and a wall with openings subjected to both vertical and horizontal loads. The reader is referred to [17] for the details of the numerical calculations.

Further applications of the LSM in the field of shape optimization problems are addressed to future works.

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APPENDIX

Let us consider arbitrary functions $\hat{\varphi} \in S_h$, $\hat{\mathbf{q}} \in T_h$ and, in correspondence with each couple of nodes n, s connected by an interface Γ_n^s of the primal mesh, the region $\hat{\Omega}_n^s$ formed by two adjacent sub-elements of $\hat{\Omega}_n$ and $\hat{\Omega}_s$ (Figure A1). Since each element $\hat{\Omega}_n^s$ recurs twice when all the nodes of the primal mesh are taken into consideration, upon applying over such elements the generalized Green formula (see, e.g., Temam [19]), we find

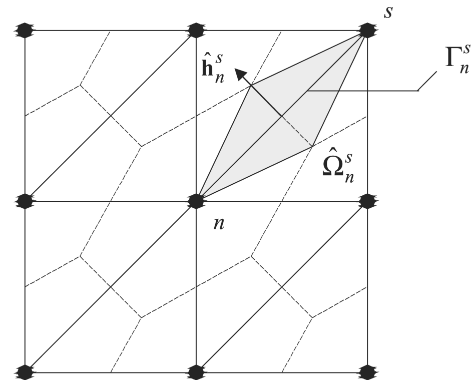

FIG. A1. Double sub-element $\hat{\Omega}_n^s$ of the dual mesh.

$$\int_\Omega div\, \hat{\mathbf{q}} \cdot \nabla \hat{\varphi}\, da = \frac{1}{2} \sum_{n=1}^{N} \sum_{s=1}^{\mathcal{S}_n} \left(- \int_{\hat{\Omega}_n^s} \hat{\mathbf{q}} \cdot \mathbf{H} \hat{\varphi}\, da + \int_{\partial \hat{\Omega}_n^s} \hat{\mathbf{q}} \cdot \nabla \hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma \right) \tag{A.1}$$

where $\mathcal{S}_n$ is the number of nodes connected to n.

In the right-hand side of Eq. (A.1), $\mathbf{H}\hat{\varphi}$ is a combination of Dirac deltas uniformly distributed along the interfaces Γ_n^s, with amplitude (per unit length) $[\![\nabla \hat{\varphi}]\!]_n^s \otimes \hat{h}_n^s$. Here, $[\![\nabla \hat{\varphi}]\!]_n^s$ is the jump of $\nabla \hat{\varphi}$ through Γ_n^s and $\hat{\mathbf{h}}_n^s$ is the unit vector orthogonal to Γ_n^s (Figure A1).

Now, said $\hat{\omega}_n^s$ and γ_n^s the intersections of $\hat{\Omega}_n^s$ and Γ_n^s with $\hat{\Omega}_n$, respectively, and said ℓ_n^s the length of Γ_n^s, it results (recall that the dual mesh divides the edges of the primal mesh in equal parts)

$$\int_{\hat{\omega}_n^s} \mathbf{H}\hat{\varphi}\, da = [\![\nabla \hat{\varphi}]\!]_n^s \otimes \hat{h}_n^s \frac{\ell_n^s}{2}, \tag{A.2}$$

$$\int_{\hat{\omega}_n^s} \mathbf{H}\hat{\varphi} \cdot \mathbf{p}\, da = [\![\nabla \hat{\varphi}]\!]_n^s \otimes \hat{h}_n^s \cdot \int_{\gamma_n^s} \mathbf{p}\, d\sigma, \;\; \forall \mathbf{p} \in \left(C\left(\hat{\omega}_n^s \right) \right)^4 . \tag{A.3}$$

From Eqs. (A.2)–(A.3), upon expressing $\hat{\mathbf{q}}$ as $\sum_{n=1}^N \hat{\mathbf{q}}(n)\, \chi_n$, we deduce

$$\int_{\hat{\Omega}_n^s} \hat{\mathbf{q}} \cdot \mathbf{H}\hat{\varphi}\, da = [\![\nabla \hat{\varphi}]\!]_n^s \otimes \hat{\mathbf{h}}_n^s \cdot \hat{\mathbf{q}}(n) \frac{\ell_n^s}{2} + [\![\nabla \hat{\varphi}]\!]_n^s \otimes \hat{\mathbf{h}}_n^s \cdot \hat{\mathbf{q}}(s) \frac{\ell_n^s}{2} = \hat{\mathbf{q}}(n) \cdot \int_{\hat{\omega}_n^s} \mathbf{H}\hat{\varphi}\, da + \hat{\mathbf{q}}(s) \cdot \int_{\hat{\omega}_s^n} \mathbf{H}\hat{\varphi}\, da \tag{A.4}$$

Still in Eq. (A1), boundary terms associated with the interfaces $\partial \hat{\Omega}_n^s$ which do not lie on $\partial \Omega$ eliminate two by two.

Moreover, for nodes n, s lying on $\partial\Omega$, it results

$$\int_{\partial\hat{\Omega}_n^s \cap \partial\Omega} \hat{\mathbf{q}} \cdot \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma = \hat{\mathbf{q}}(n) \cdot \int_{\gamma_n^s} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma + \hat{\mathbf{q}}(s) \cdot \int_{\gamma_s^n} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma. \tag{A.5}$$

Thus, formula (A.1) can be reduced to

$$\int_\Omega \nabla\, \hat{\varphi} \cdot div\, \hat{\mathbf{q}}\, da = -\sum_{n=1}^{N} \hat{\mathbf{q}}(n) \cdot \int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da + \sum_{b\in B} \hat{\mathbf{q}}(b) \cdot \int_{\gamma_b} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma, \tag{A.6}$$

where $\gamma_b = \partial\hat{\Omega}_b \cap \partial\Omega$, $\forall b \in B$.

Consider now $\vartheta_h\hat{\mathbf{q}} \in (S_h)^4$ defined as in Eq. (44). For such a function and an arbitrary $\hat{\varphi} \in S_h$, we obtain

$$\int_\Omega div\, \vartheta_h\, \hat{\mathbf{q}} \cdot \nabla\hat{\varphi}\, da = \frac{1}{2}\sum_{n=1}^{N}\sum_{s=1}^{S_n}\left(-\int_{\hat{\Omega}_n^s} \vartheta_h\hat{\mathbf{q}} \cdot \mathbf{H}\hat{\varphi}\, da + \int_{\partial\hat{\Omega}_n^s} \vartheta_h\hat{\mathbf{q}} \cdot \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma\right), \tag{A.7}$$

$$\begin{aligned}\int_{\hat{\Omega}_n^s} \vartheta_h\hat{\mathbf{q}} \cdot \mathbf{H}\hat{\varphi}\, da &= [\![\nabla\hat{\varphi}]\!]_n^s \otimes \hat{\mathbf{h}}_n^s \cdot \left(\hat{\mathbf{q}}(n)\frac{\ell_n^s}{2} + (\hat{\mathbf{q}}(s) - \hat{\mathbf{q}}(n))\frac{\ell_n^s}{8}\right) \\ &\quad + [\![\nabla\hat{\varphi}]\!]_n^s \otimes \hat{\mathbf{h}}_n^s \cdot \left(\hat{\mathbf{q}}(s)\frac{\ell_n^s}{2} + (\hat{\mathbf{q}}(n) - \hat{\mathbf{q}}(s))\frac{\ell_n^s}{8}\right) \\ &= \hat{\mathbf{q}}(n) \cdot \int_{\hat{\omega}_n^s} \mathbf{H}\hat{\varphi}\, da + \hat{\mathbf{q}}(s) \cdot \int_{\hat{\omega}_s^n} \mathbf{H}\hat{\varphi}\, da.\end{aligned} \tag{A.8}$$

Further on, for nodes n, s lying on $\partial\Omega$, one gets

$$\begin{aligned}\int_{\partial\hat{\Omega}_n^s \cap \partial\Omega} \vartheta_h\hat{\mathbf{q}} \cdot \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma &= \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}|_{\gamma_n^s} \cdot \left(\hat{\mathbf{q}}(n)\frac{\ell_n^s}{2} + (\hat{\mathbf{q}}(s) - \hat{\mathbf{q}}(n))\frac{\ell_n^s}{8}\right) \\ &\quad + \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}|_{\gamma_s^n} \cdot \left(\hat{\mathbf{q}}(s)\frac{\ell_n^s}{2} + (\hat{\mathbf{q}}(n) - \hat{\mathbf{q}}(s))\frac{\ell_n^s}{8}\right) \\ &= \hat{\mathbf{q}}(n) \cdot \int_{\gamma_n^s} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma + \hat{\mathbf{q}}(s) \cdot \int_{\gamma_s^n} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma.\end{aligned} \tag{A.9}$$

Upon substituting Eqs. (A.8)–(A.9) into Eq. (A.7), we find

$$\begin{aligned}\int_\Omega \nabla\, \hat{\varphi} \cdot div\, \vartheta_h\hat{\mathbf{q}}\, da &= -\sum_{n=1}^{N} \hat{\mathbf{q}}(n) \cdot \int_{\hat{\Omega}_n} \mathbf{H}\hat{\varphi}\, da + \sum_{b\in B} \hat{\mathbf{q}}(b) \cdot \int_{\gamma_b} \nabla\hat{\varphi} \otimes \hat{\mathbf{n}}\, d\sigma \\ &= \int_\Omega \nabla\, \hat{\varphi} \cdot div\, \hat{\mathbf{q}}\, da.\end{aligned} \tag{A.10}$$

Now, let us adopt the following expansion of $\vartheta_h\hat{\mathbf{q}}(\mathbf{x})$ over the generic dual element $\hat{\Omega}_n$

$$\vartheta_h\hat{\mathbf{q}}(\mathbf{x}) = \hat{\mathbf{q}}(n) + \nabla\vartheta_h\hat{\mathbf{q}}(\mathbf{x})(\mathbf{x} - \mathbf{x}_n), \quad \forall \mathbf{x} \in \hat{\Omega}_n. \tag{A.11}$$

Formula (A.11) leads us to deduce

$$\int_\Omega \vartheta_h\hat{\mathbf{q}} \cdot \hat{\boldsymbol{\psi}}\, da = \int_\Omega \hat{\mathbf{q}} \cdot \hat{\boldsymbol{\psi}}\, da + \mathrm{O}(\hat{\boldsymbol{\psi}},\, \vartheta_h\hat{\mathbf{q}}), \quad \forall\hat{\boldsymbol{\psi}} \in T_h, \tag{A.12}$$

where

$$\left|\mathrm{O}(\hat{\boldsymbol{\psi}},\, \vartheta_h\hat{\mathbf{q}})\right| \le h\|\hat{\boldsymbol{\psi}}\|_0\, |\vartheta_h\hat{\mathbf{q}}|_1\,. \tag{A.13}$$

From Eqs. (A.6), (A.10), and (A.12)–(A.13), we get the proof of formulas (36), (45)–(46), and (69) of the present paper.